$K \to \pi$ and $K \to 0$ in 2+1 Flavor Partially Quenched Chiral Perturbation Theory

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Abstract

We calculate results for $K \to \pi$ and $K \to 0$ matrix elements to next-to-leading order in 2+1 flavor partially quenched chiral perturbation theory. Results are presented for both the $\Delta I = 1/2$ and $3/2$ channels, for chiral operators corresponding to current-current, gluonic penguin, and electroweak penguin 4-quark operators. These formulas are useful for studying the chiral behavior of currently available 2+1 flavor lattice QCD results, from which the low energy constants of the chiral effective theory can be determined. The low energy constants of these matrix elements are necessary for an understanding of the $\Delta I = 1/2$ rule, and for calculations of ϵ'/ϵ using current lattice QCD simulations.

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I. INTRODUCTION

Lattice QCD is a first principles approach to calculating low energy hadronic quantities using numerical Monte Carlo methods. State-of-the-art calculations are now including 2+1 flavors of quarks in the weighting of the gauge configurations, thus eliminating the quenched approximation. However, partially quenched simulations, where the valence quarks have different masses than those of the sea quarks, are still of use when combined with partially quenched chiral perturbation theory (PQχPT) [1]. Since chiral perturbation theory (χPT) comes with a number of unknown low energy constants (LEC's), these LEC's must be obtained from non-perturbative methods, e.g., lattice calculations, or from experiment, in order to have predictive power. When the number of light sea quarks is equal to three, then the LEC's of PQχPT correspond to those of the unitary theory [2, 3], and the LEC's obtained from fits to partially quenched lattice data can be used to predict hadronic quantities. Partial quenching can therefore be used in order to gain a better handle on chiral fits to numerical data, because varying the sea and valence quark masses separately leads to the determination of more linearly independent combinations of LEC's. It also allows one to make use of more of the available lattice data, since simulating additional valence quark masses is relatively cheap compared to generating more ensembles with different sea quark masses.

In this work we calculate PQχPT expressions relevant for obtaining $K \to \pi\pi$ matrix elements from lattice simulations. Although matrix elements of $K \to \pi\pi$ are of importance to phenomenology, there are difficulties with extracting multi-hadron decay amplitudes directly from the lattice, as expressed by the Maiani-Testa no-go theorem [4]. The implication of this no-go theorem is that physical amplitudes can only be computed if the final state pions are at rest, or some other unphysical set of kinematics. It was shown by Lellouch and Lüscher [5] (see also Ref. [6]) that this no-go theorem can be evaded, and that the matrix elements can be computed at physical kinematics using finite volume correlation functions. Although this method does not require χPT, the physical volume necessary to implement the method at physical quark masses is large, and therefore prohibitively expensive given the present computational resources.

An alternative method for calculating $K \to \pi\pi$ from lattice QCD simulations is to obtain the leading order LEC's necessary to construct $K \to \pi\pi$ from lattice simulations of the simpler quantities $K \to \pi$ and $K \to 0$. This method was introduced quite some time ago

in Ref. [7]. Given that there are large corrections to kaon matrix elements coming from chiral logarithms at higher orders in $SU(3)$ χPT, it is necessary to include next-to-leading order (NLO) corrections in the fits to lattice data. This is true both because the light quark masses are still relatively heavy in present simulations, and also the physical strange quark mass is itself rather heavy. It is an important, and as yet unanswered question whether the kaon mass is light enough so that $K \to \pi\pi$ amplitudes can be described by one-loop chiral perturbation theory to a useful precision. The issue of convergence is quantity dependent, and so must be studied for each quantity of interest. We thus calculate the NLO PQχPT expressions for $K \to \pi$ and $K \to 0$ matrix elements, including finite-size effects, which are needed both to extract LEC's from the lattice, and to assess the convergence of χPT by studying fits to lattice data as a function of quark masses.

In this work we calculate PQχPT $K \to \pi$ and $K \to 0$ matrix elements in the isospin (2+1-flavor) limit. We do not consider the completely non-degenerate quark mass case since isospin breaking leads to additional complications [such as (8,1)'s contributing to $\Delta I = 3/2$ amplitudes], and these would also not be relevant to current lattice simulations. Thus, we restrict ourselves to the 2+1 case in both the sea and valence sectors, but with no degeneracies between sea and valence quark masses. We do not present here a complete set of formulas necessary to extract all of the NLO LEC's from 2+1 flavor lattice calculations, since some of the needed LEC's must be obtained from $K \to \pi\pi$ amplitudes at unphysical kinematics. Even so, the formulas should be useful in extracting leading order LEC's from lattice data, and in studying the convergence of the chiral expansion. Note that there are many works which discuss the determination of the LEC's needed to construct $K \to \pi\pi$ through NLO in χPT at physical kinematics [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18], though we make no attempt to review the various approaches here.

For the (8,1) ($\Delta I = 1/2$) amplitudes there is an additional complication in the partially quenched theory coming from the treatment of the gluonic penguin 4-quark operator. For the three-flavor theory the situation in PQχPT is simplified significantly if the corresponding chiral operators are chosen to transform as (8,1)'s under the partially quenched graded symmetry group [12, 19]. That is the prescription we adopt in the current work. If another choice is made, such as, for example, if the chiral operators are chosen to transform under the (8,1) chiral symmetry group of the full theory, then additional LEC's enter the calculation, making the determination of the desired LEC's more complicated. Although this

complication requires some care in the three flavor partially quenched theory, the method is still viable, unlike the quenched theory, in which quenched gluonic penguin amplitudes lead to large systematic uncertainties [12, 19, 20, 21, 22].

This paper is organized as follows: in Sec. II we give a review of PQχPT, including the effects of the weak Lagrangian, and in Sec. III we give a quick overview of the calculation involved. In Sec. IV we review the operator subtraction that is necessary for $\Delta I = 1/2$ matrix elements, and introduce the $\Theta^{(3,\overline{3})}$ operator for this purpose. NLO formulas of matrix elements of this operator are calculated for use in later sections. We present results for the (8,8) electroweak penguin operators for the $K \to 0$ and $K \to \pi$ processes in Sec. V, where we also give the physical $K \to \pi\pi$ amplitudes for completeness. In Sec. VI we present results for the (27,1), $\Delta I = 3/2$, $K \to \pi$ matrix element, and in Sec. VII we present the results for (8,1) and (8,1)+(27,1) operators for $K \to 0$ and $K \to \pi$, including the operator subtraction. In Sec. VIII we discuss the finite volume corrections for the results presented in this work. We conclude in Sec. IX and include relevant function definitions and the chiral logarithm contributions in a set of appendices. Appendix H provides an erratum for Refs. [10] and [11].

II. PARTIALLY QUENCHED CHIRAL PERTURBATION THEORY

We use the standard formulation of partially quenched chiral perturbation theory (PQχPT) introduced in [23, 24]. In this formulation, the valence quark loops are removed by introducing "ghost" quarks with the same masses and quantum numbers as their valence counterparts, but which obey opposite statistics. The chiral symmetry group for a partially quenched theory is graded; in general one takes it to be $SU\left(N_{\text{val}} + N_{\text{sea}} \,|\, N_{\text{val}}\right)_L \otimes SU\left(N_{\text{val}} + N_{\text{sea}} \,|\, N_{\text{val}}\right)_R$. For the purposes of this work, we set $N_{\text{val}} = N_{\text{sea}} = 3$. Specifically, we have three valence quarks denoted as x, y, and z; three sea quarks denoted as u, d, and s; and finally three ghosts: $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$.

A. Strong Lagrangian in PQχPT

As explained in Ref. [11], in the partially quenched theory, operators are written in terms of the chiral field

$$\Sigma = \exp\left[\frac{2i\Phi}{f}\right] , \tag{1}$$

where f is the meson decay constant in the $SU(3)$ chiral limit (normalized such that the physical $f_\pi \approx 130.7$ MeV), Φ is a 9×9 matrix containing the meson fields,

$$\Phi \equiv \begin{pmatrix} \phi & \chi^\dagger \\ \chi & \tilde{\phi} \end{pmatrix} , \tag{2}$$

where ϕ is a 6×6 matrix of pseudoscalar mesons constructed out of valence and sea quarks, $\tilde{\phi}$ is a 3×3 matrix containing mesons constructed with two ghost quarks, χ ($\chi^\dagger$) is a 3×6 (6×3) matrix containing fermionic mesons made out of one quark and one ghost quark. Σ transforms under the graded chiral symmetry group as

$$\Sigma \to L\Sigma R^\dagger , \tag{3}$$

with $L \in SU(6\,|\,3)_L\,, R \in SU(6\,|\,3)_R$. Operators in the chiral effective theory are constructed from the quark-level operators out of Σ and other objects (such as the quark charge matrix and mass matrix, for example) such that they transform the same way under the chiral symmetry group.

The leading-order (LO) strong Lagrangian is given by [25]

$$\mathcal{L}_{st}^{(2)} = \frac{f^2}{8}\mathrm{str}\left[\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right] + \frac{f^2 B_0}{4}\mathrm{str}\left[\mathcal{M}\Sigma + \Sigma^\dagger\mathcal{M}^\dagger\right] , \tag{4}$$

where the superscript (2) indicates that this Lagrangian is valid to $\mathcal{O}(p^2)$ in the chiral power counting scheme, and $\mathcal{M}$ is the quark mass matrix

$$\mathcal{M} = \mathrm{diag}\left(m_x, m_y, m_z, m_u, m_d, m_s, m_x, m_y, m_z\right) . \tag{5}$$

Note that this corresponds to the quark vector composed of valence quarks, sea quarks, and ghost quarks

$$q = (x, y, z, u, d, s, \tilde{x}, \tilde{y}, \tilde{z})^{\mathrm{T}} . \tag{6}$$

The supertrace is defined as follows: for a 9×9 matrix

$$U_{9\times9} = \begin{pmatrix} A_{6\times6} & B_{6\times3} \\ C_{3\times6} & D_{3\times3} \end{pmatrix} \tag{7}$$

in which sub-matrix A is the top-left 6×6 diagonal block and D is the bottom-right 3×3 diagonal block, then

$$\mathrm{str}\,(U) = \mathrm{tr}\,(A) - \mathrm{tr}\,(D)\,. \tag{8}$$

We set the valence x and y quark masses equal, and we set the sea u and d quark masses equal,

$$m_x = m_y, \quad m_u = m_d. \tag{9}$$

Thus we work in the isospin limit in both the valence and sea sector, and we present results for both this (2+1-flavor) case and the 3- flavor case (degenerate valence quarks).

At NLO in the full theory [$\mathcal{O}\left(p^4\right)$], the strong Lagrangian involves 12 additional operators with undetermined coefficients [25, 26]. There is an additional $\mathcal{O}(p^4)$ operator which appears in the partially quenched theory [27], though this operator does not contribute to the quantities considered in this work. The NLO operators of the strong Lagrangian relevant for the current work are

$$\begin{aligned}
\mathcal{O}_4^{(st)} &= \mathrm{str}\left[L^2\right]\mathrm{str}\left[S\right], \\
\mathcal{O}_5^{(st)} &= \mathrm{str}\left[L^2 S\right], \\
\mathcal{O}_6^{(st)} &= \mathrm{str}\left[S\right]^2, \\
\mathcal{O}_8^{(st)} &= \frac{1}{2}\mathrm{str}\left[S^2 - P^2\right]\ ,
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
S &= 2B_0\left(\mathcal{M}^\dagger\Sigma^\dagger + \Sigma\mathcal{M}\right), \\
P &= 2B_0\left(\mathcal{M}^\dagger\Sigma^\dagger - \Sigma\mathcal{M}\right), \\
L_\mu &= i\Sigma\partial_\mu\Sigma^\dagger.
\end{aligned} \tag{11}$$

As follows from the strong Lagrangian above, the leading-order mass of a bare pseudoscalar meson is

$$m_{ij}^2 = B_0\left(m_i + m_j\right)\ , \tag{12}$$

where m_{ij} is the mass of meson Φ_{ij}, m_i and m_j are the masses of the quarks q_i and q_j (i, j can refer to the sea, valence, or ghost quarks in this case). In our partially quenched amplitudes we assume that the light quark masses are all light enough compared to the η' mass so that the η' can be integrated out. As demonstrated in Ref. [3], this is the case where

the LEC's of the partially quenched theory with three sea quarks correspond to the LEC's of the unitary theory.

In the following we adopt the notation that the masses of mesons which are constructed out of two different flavors of quarks are labelled in terms of their quark consituents, regardless of whether they are sea or valence, for example m_{xy} or m_{zs}. For any flavor-neutral meson, we use $m_D \equiv m_{dd}$ or $m_X \equiv m_{xx}$ for mesons in the "flavor basis." Due to the disconnected propagators which arise in the flavor-neutral sector, this is distinct from the "physical basis," where the relevant mesons are the π^0 and η. These only arise in the sea sector, and we will use the fact that

$$\begin{aligned} m_{\pi^0}^2 &= m_D^2 \ , \\ m_\eta^2 &= \frac{1}{3}\left(2m_S^2 + m_D^2\right) , \end{aligned}$$

in the isospin limit ($m_u = m_d$).

The propagators for flavor neutral mesons are obtained by following the prescription in Ref.[3] in Minkowski space:

$$\overset{\sqcap}{\Phi_{ii}\Phi_{jj}} = \frac{i\delta_{ij}\varepsilon_i}{p^2 - m_{ii}^2 + i\epsilon} - \frac{i}{3}\frac{\left(p^2 - m_D^2\right)\left(p^2 - m_S^2\right)}{\left(p^2 - m_{ii}^2\right)\left(p^2 - m_{jj}^2\right)\left(p^2 - m_\eta^2\right)} \ . \tag{13}$$

The propagators for flavor off-diagonal mesons are

$$\overset{\sqcap}{\Phi_{ij}\Phi_{ji}} = \frac{i\varepsilon_j}{p^2 - m_{ij}^2 + i\epsilon} \tag{14}$$

where

$$\varepsilon_j = \begin{cases} 1 & j \in \{x, y, z, u, d, s\} \\ -1 & j \in \{\tilde{x}, \tilde{y}, \tilde{z}\} \end{cases} \ . \tag{15}$$

B. Leading-Order Weak Lagrangian

In full χPT, we group the weak operators appearing in the $K \to \pi\pi$ transition by their chiral transformation properties in the $SU(3)_L \otimes SU(3)_R$ symmetry group. We can carry this same idea over to PQχPT, where we extend their definition into the graded group using the expanded chiral field Σ and replacing traces with supertraces. Except for those cases discussed explicitly (such as the quark charge and mass matrices, for example), when going

from unquenched to partially quenched χPT, operators are replaced as follows:

$$\lambda \rightarrow \begin{pmatrix} \lambda_{3\times3} & 0_{3\times6} \\ 0_{6\times3} & 0_{6\times6} \end{pmatrix} , \tag{16}$$

where the upper left block of this matrix is the 3×3 block corrsponding to the valence sector.

The leading order weak operators are [7, 8, 9, 11, 28]

$$\begin{aligned} \mathcal{O}_{LO}^{(8,8)} &= \mathrm{str}\left[\lambda_6 \Sigma Q \Sigma^\dagger\right] \\ \mathcal{O}_{LO,1}^{(8,1)} &= \mathrm{str}\left[\lambda_6 \partial_\mu \Sigma \partial^\mu \Sigma^\dagger\right] \\ \mathcal{O}_{LO,2}^{(8,1)} &= 2B_0 \mathrm{str}\left[\lambda_6 \left(\Sigma \mathcal{M} + \mathcal{M}^\dagger \Sigma^\dagger\right)\right] \\ \mathcal{O}_{LO}^{(27,1)} &= t^{ij}_{kl} \left(\Sigma \partial_\mu \Sigma^\dagger\right)^k_i \left(\Sigma \partial^\mu \Sigma^\dagger\right)^l_j \end{aligned} \tag{17}$$

where Q is the quark charge matrix, $(\lambda_6)_{ij} = \delta_{i3}\delta_{j2}$, and the tensor t^{ij}_{kl} is symmetric on any indices and traceless on pairs of upper and lower indices, and its elements are chosen to pick out the $\Delta S = 1$ transitions; it thus plays a similar role to that of λ_6 for the other operators. However, we will defer the actual determination of its non-zero elements until Section VI A, where we will use this tensor to further divide the operator into the isospin 3/2 part and the isospin 1/2 part and directly evaluate their respective amplitudes. The isospin decomposition of $K \rightarrow \pi$ matrix elements is given in Appendix A.

There is a choice to be made for the quark charge matrix, Q, above, which enters in the electroweak penguin operators [11]. We could either assign charges to ghosts such that they cancel out the electroweak valence quark loops, or we could make them uncharged. In this paper we derive the amplitudes with the electroweak penguin operators for both choices, which we denote as Q_1 and Q_2. We always assign zero charge to the sea quarks, since that is what is typically done when generating lattice gauge fields. The two choices of charge matrix are:

$$\begin{aligned} Q_1 &= \mathrm{diag}\left(2,-1,-1,0,0,0,2,-1,-1\right), \\ Q_2 &= \mathrm{diag}\left(2,-1,-1,0,0,0,0,0,0\right). \end{aligned} \tag{18}$$

We discuss the weak operators which contribute to $K \rightarrow \pi$ and $K \rightarrow \pi\pi$ at next-to-leading order in subsequent sections.

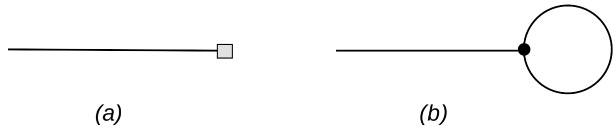

Figure 1: Diagrams contributing to $K \to 0$ at NLO. The gray square is the insertion of a NLO weak vertex, and the small dot is an insertion of the LO weak vertex.

III. DETAILS OF THE CALCULATION

To make complete use of lattice data in extracting LEC's relevant for $K \to \pi\pi$, it is important to work in the non-degenerate $m_x = m_y \neq m_z$ case. Since the $K \to \pi$ amplitudes do not conserve 4-momentum for $m_z \neq m_x$, the weak operator must transfer a 4 momentum $q \equiv p_{xz} - p_X$. In our calculations we restrict ourselves to the case where both initial and final mesons are at rest, so $q = (m_{xz} - m_X, 0, 0, 0)$.

The NLO diagrams contributing to $K \to 0$ and $K \to \pi$ are given in Figs. 1 and 2, respectively. The external legs are always mesons made of two valence quarks, while the internal loops in the partially-quenched theory consist of valence-ghost, valence-sea, and valence-valence mesons. In addition to these diagrams, the renormalization of the external legs (wave-function renormalization) via the strong interactions must be taken into account.

The logarithmic expressions presented in the appendices of this work are quite lengthy. Thus, checks are necessary. The first check was that the one-loop insertions cancel those of the divergent counterterms, and this check was performed for all expressions in this paper. Another check is that an expression reduces to some other in the appropriate limit. All of the logarithmic expressions in this paper reduce to those in Refs. [8, 9, 10, 11, 18] in the appropriate degenerate sea quark and full QCD limits.[1] Finally, all one-loop expressions in this work were computed separately by at least two of the authors, using independently written code. The codes used were the FEYNCALC package [29] written for the Mathematica [30] system, and the FORMCalc package [31], which interfaces FORM [32] with Mathematica.

[1] Note that Ref. [8] contains errors that are corrected in Ref. [9], and we agree with the latter.

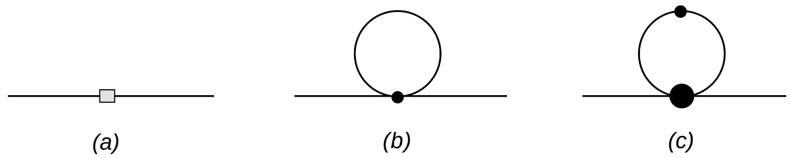

Figure 2: Diagrams contributing to $K \to \pi$ at NLO. As in Fig. 1, the gray square is an insertion of a NLO weak vertex, and the small dot is an insertion of the LO weak vertex. The large dot is the insertion of an $O(p^2)$ strong vertex.

IV. SUBTRACTION OF $\Delta I = 1/2$ AMPLITUDES

In general the $\Delta I = 1/2$ matrix elements of four-quark operators have a power divergent part due to the four-quark operators mixing under renormalization with lower dimensional operators when using a lattice regularization. This power divergence reduces to a quark bilinear times a momentum independent coefficient [28]. Following Refs. [7, 11, 28], in order to remove the power-divergence of $\Delta I = 1/2$ operators, we perform a subtraction using the dimension three quark-level operator,

$$\Theta^{(3,\bar{3})} \equiv \bar{s}\,(1-\gamma_5)\,d\ . \tag{19}$$

This subtraction must also be performed in PQχPT for comparison with the subtracted lattice results. Again following Refs. [7, 11, 28], the lowest order $[\mathcal{O}(p^0)]$ chiral operator corresponding to the $(3,\bar{3})$ operator in Eq. (19) is

$$\Theta^{(3,\bar{3})}_{LO} = \alpha^{(3,\bar{3})} \mathrm{str}\left[\lambda_6 \Sigma\right]\ , \tag{20}$$

where the low-energy constant $\alpha^{(3,\bar{3})}$ can be related to the coefficient of the mass term in the leading-order strong Lagrangian [11],

$$\alpha^{(3,\bar{3})} = -\frac{f^2 B_0}{2}\ .$$

As explained in Ref [28], the $\Theta^{(3,\bar{3})}$ operator can be used to remove the power divergences to all orders in the lattice calculation. This subtraction is performed to NLO in PQχPT explicitly in the sections that follow. To this end, we require the higher order chiral operators of $\Theta^{(3,\bar{3})}$. The terms up to $O(p^2)$ needed for this work are

$$\Theta^{(3,\bar{3})} = \Theta^{(3,\bar{3})}_{LO} + \sum_i c_{33,i}\mathcal{O}'_i \tag{21}$$

where i takes the values $4, 5, 6, 8, H_2$, and where

$$
\begin{aligned}
\mathcal{O}'_4 &= \frac{1}{2}\mathrm{str}\left[\lambda_6\Sigma\right]\mathrm{str}\left[\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\right], \\
\mathcal{O}'_5 &= \frac{1}{2}\mathrm{str}\left[\lambda_6\Sigma\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\right], \\
\mathcal{O}'_6 &= 2B_0\mathrm{str}\left[\lambda_6\Sigma\right]\mathrm{str}\left[\mathcal{M}^\dagger\Sigma+\Sigma^\dagger\mathcal{M}\right], \\
\mathcal{O}'_8 &= 2B_0\mathrm{str}\left[\lambda_6\Sigma\mathcal{M}^\dagger\Sigma\right], \\
\mathcal{O}'_{H_2} &= B_0\mathrm{str}\left[\lambda_6\mathcal{M}\right].
\end{aligned}
\tag{22}
$$

The coefficients $c_{33,i}$ of the operators $\mathcal{O}'_i$ are related to the Gasser-Leutwyler coefficients by $c_{33,i} = -8B_0L_i$, a relation similar to that for the leading order coefficient, $\alpha^{(3,\bar{3})}$, given above.

To NLO, the $K \to 0$ matrix element for 2+1 valence flavors is

$$
\begin{aligned}
\left\langle 0\left|\Theta^{(3,\bar{3})}\right|K^0\right\rangle &= \frac{2i}{f}\alpha^{(3,\bar{3})}\left[1+\frac{1}{2}\delta Z_{xz}\right] \\
&+\frac{4i}{9}\frac{\alpha^{(3,\bar{3})}}{f^3}\Bigg\{\Big[1+R_X(m_\eta,m_Z)-R_\eta(m_X,m_X)\Big]\ell(m_X^2) \\
&+\Big[1+R_Z(m_\eta,m_X)-R_\eta(m_Z,m_Z)\Big]\ell(m_Z^2) \\
&-6\ell(m_{xd}^2)-3\ell(m_{xs}^2)-6\ell(m_{zd}^2)-3\ell(m_{zs}^2) \\
&+\Big[R_\eta(m_X,m_X)+R_\eta(m_X,m_Z)+R_\eta(m_Z,m_Z)\Big]\ell(m_\eta^2) \\
&-R_X(m_\eta)\tilde{\ell}(m_X^2)-R_Z(m_\eta)\tilde{\ell}(m_Z^2)\Bigg\} \\
&-\frac{32iB_0}{f}\left\{L_8m_{xz}^2+L_6\left(2m_D^2+m_S^2\right)\right\},
\end{aligned}
\tag{23}
$$

where the chiral logarithms $\ell(m^2)$ and $\tilde{\ell}(m^2)$ are defined in Appendix B, along with the residues $R_x(m_a)$, $R_x(m_a, m_b)$. The wave-function renormalization δZ_{xz} is given in Appendix C.

To NLO, the $K \to \pi$ matrix element (also for 2+1 valence flavors) is

$$
\begin{aligned}
\left\langle \pi^+\left|\Theta^{(3,\bar{3})}\right|K^+\right\rangle &= -\frac{2}{f^2}\alpha^{(3,\bar{3})}-\frac{16B_0}{f^2}\Big\{L_5m_Xm_{xz}-2L_8\left(m_X^2+m_{xz}^2\right) \\
&-2L_6\left(2m_D^2+m_S^2\right)\Big\}+\left\langle \pi^+\left|\Theta^{(3,\bar{3})}\right|K^+\right\rangle_{\mathrm{logs}}
\end{aligned}
\tag{24}
$$

For clarity, the rather lengthy logarithmic contribution is given in Appendix D. For degen-

erate valence masses ($m_x = m_y = m_z$), the $K \to \pi$ amplitude simplifies to

$$
\begin{aligned}
\left\langle \pi^+ \left| \Theta^{(3,\bar{3})} \right| K^+ \right\rangle^{\text{deg.val.}} = &- \frac{2}{f^2}\alpha^{(3,\bar{3})}\left[1 + \delta Z_X\right] \\
&+ \frac{4}{3}\frac{\alpha^{(3,\bar{3})}}{f^4}\Bigg\{ m_X^2 \left[R_X(m_\eta)\tilde{\tilde{\ell}}(m_X^2) - 2R_\eta(m_X, m_X)\beta(0, m_\eta^2, m_X^2)\right] \\
&+ \left[-1 + R_\eta(m_X, m_X)\right]\ell(m_X^2) + 2\ell(m_{xd}^2) + \ell(m_{xs}^2) \\
&- R_\eta(m_X, m_X)\ell(m_\eta^2) \\
&+ \left[2m_X^2\Big(1 - R_\eta(m_X, m_X)\Big) + R_X(m_\eta)\right]\tilde{\ell}(m_X^2)\Bigg\} \\
&- \frac{16B_0}{f^2}\Big\{\left(L_5 - 4L_8\right)m_X^2 - 2L_6\left(2m_D^2 + m_S^2\right)\Big\},
\end{aligned}
\tag{25}
$$

where $\tilde{\tilde{\ell}}(m^2)$ and $\beta(q^2, m_1^2, m_2^2)$ are defined in Appendix B, and δZ_X is given in Appendix C. These expressions are used below when performing the power divergent operator subtractions that are necessary in order to obtain the physical amplitudes in which we are interested.

V. WEAK MATRIX ELEMENTS WITH $(8, 8)$, $\Delta I = 3/2$ AND $1/2$ OPERATORS

In this section we present the results for the chiral operators which transform as (8,8)'s under the chiral symmetry. These correspond to the electroweak penguin 4-quark operators. Formulas are presented for $K \to 0$ and $K \to \pi$ for nondegenerate ($m_x = m_y \neq m_z$) valence quark masses, as well as for $K \to \pi$ with degenerate valence quark masses. The power divergent subtraction is discussed for the $\Delta I = 1/2$, $K \to \pi$ amplitude. Since $K \to 0$ and $K \to \pi$ are sufficient to construct $K \to \pi\pi$ to NLO at physical kinematics for the (8,8)'s, we present the physical $K \to \pi\pi$ amplitudes as well.

Following Ref. [11], the form of the operator $\mathcal{O}^{(8,8)}$ through NLO in PQχPT is

$$\begin{aligned}\mathcal{O}^{(8,8)} =& \alpha_{88}\mathrm{str}\left[\lambda_6\Sigma Q\Sigma^\dagger\right] \\ &+ c_{88,1}\mathrm{str}\left[\lambda_6 L_\mu \Sigma Q\Sigma^\dagger L^\mu\right] \\ &+ c_{88,2}\mathrm{str}\left[\lambda_6 L_\mu\right]\mathrm{str}\left[\Sigma Q\Sigma^\dagger L^\mu\right] \\ &+ c_{88,3}\mathrm{str}\left[\lambda_6\left\{\Sigma Q\Sigma^\dagger, L^2\right\}\right] \\ &+ c_{88,4}\mathrm{str}\left[\lambda_6\left\{\Sigma Q\Sigma^\dagger, S\right\}\right] \\ &+ c_{88,5}\mathrm{str}\left[\lambda_6\left[\Sigma Q\Sigma^\dagger, P\right]\right] \\ &+ c_{88,6}\mathrm{str}\left[\lambda_6\Sigma Q\Sigma^\dagger\right]\mathrm{str}\left[S\right] \ , \end{aligned} \tag{26}$$

with S, P, and L_μ as defined in Eq. (11).

A. $K \to 0$ amplitudes for 2+1 valence flavors

As explained in Section II B, there are two choices for the quark charge matrix for the operators in the (8,8) representation. If we set $Q = Q_1$, we obtain for the $K \to 0$ amplitude

$$\begin{aligned}\left\langle 0\left|\mathcal{O}^{(8,8)}\right| K^0\right\rangle_{Q_1} =& \frac{4i}{f^3}\alpha_{88}\left[-2\ell\left(m_{xd}^2\right) - \ell\left(m_{xs}^2\right) + 2\ell\left(m_{zd}^2\right) + \ell\left(m_{zs}^2\right)\right] \\ &- \frac{8i}{f}c_{88,4}\left(m_{xz}^2 - m_X^2\right) \ . \end{aligned} \tag{27}$$

If we set $Q = Q_2$, we obtain

$$\begin{aligned}\left\langle 0\left|\mathcal{O}^{(8,8)}\right| K^0\right\rangle_{Q_2} =& \frac{4i}{f^3}\alpha_{88}\Big\{-\ell\left(m_X^2\right) + 2\ell\left(m_{xz}^2\right) - \ell\left(m_Z^2\right) - 2\ell\left(m_{xd}^2\right) - \ell\left(m_{xs}^2\right) \\ &+ 2\ell\left(m_{zd}^2\right) + \ell\left(m_{zs}^2\right)\Big\} - \frac{8i}{f}c_{88,4}\left(m_{xz}^2 - m_X^2\right) \ . \end{aligned} \tag{28}$$

B. $K \to \pi$ amplitudes for 2+1 valence flavors

The process $K \to \pi$ must be separated into its $\Delta I = 3/2$ and $\Delta I = 1/2$ pieces, and we give the explicit isospin decomposition in Appendix A. For the $\Delta I = 3/2$ amplitudes we

have

$$
\begin{aligned}
\left\langle\pi^{+}\left|\mathcal{O}^{(8,8)(3/2)}\right| K^{+}\right\rangle_{Q_2} &= \left\langle\pi^{+}\left|\mathcal{O}^{(8,8)(3/2)}\right| K^{+}\right\rangle_{Q_1} \\
&= \frac{4\alpha_{88}}{f^2}+\frac{4}{f^2}\Big\{-\left(c_{88,1}+c_{88,2}\right) m_{xz} m_X \\
&\quad +2\left(c_{88,4}+c_{88,5}\right)\left(m_{xz}^2+m_X^2\right)+2 c_{88,6}\left(2 m_D^2+m_S^2\right)\Big\} \\
&\quad +\left\langle\pi^{+}\left|\mathcal{O}^{(8,8)(3/2)}\right| K^{+}\right\rangle_{Q_1,\mathrm{logs}} .
\end{aligned} \tag{29}
$$

For brevity, we give only the analytic part of these matrix elements here; the logarithmic contributions are given in Appendix E.

For the $\Delta I=1/2$ amplitudes, we are ultimately interested in the subtracted versions, as discussed in Sec. IV. We expand the amplitude $\left\langle 0\left|\Theta^{(3,\bar{3})}\right| K^0\right\rangle$ to leading nontrivial order, and take the ratio

$$
\frac{\left\langle 0\left|\mathcal{O}^{(8,8)}\right| K^0\right\rangle}{\left\langle 0\left|\Theta^{(3,\bar{3})}\right| K^0\right\rangle}=-\frac{4 c_{88,4}}{\alpha^{(3,\bar{3})}} B_0\left(m_z-m_x\right)+\frac{2}{f^2} \frac{\alpha_{88}}{\alpha^{(3,\bar{3})}}(\mathrm{logs})+\ldots \tag{30}
$$

where higher order terms in chiral perturbation theory are omitted. The power divergent contribution is proportional to m_z-m_x, and this is true to all orders in the chiral expansion by CPS symmetry [33]. Thus, the ratio of LEC's containing the power divergence, $-4c_{88,4}B_0/(\alpha^{(3,\bar{3})})$, can be extracted from the corresponding lattice matrix elements, since the mass dependence of the divergent piece is known to all orders of the chiral expansion. We perform the operator subtraction using this ratio and the amplitude $\left\langle\pi^{+}\left|\Theta^{(3,\bar{3})}\right| K^{+}\right\rangle$,

$$
\begin{aligned}
\left\langle\pi^{+}\left|\mathcal{O}_{\mathrm{sub}}^{(8,8)(1/2)}\right| K^{+}\right\rangle_Q &= \left\langle\pi^{+}\left|\mathcal{O}^{(8,8)(1/2)}\right| K^{+}\right\rangle_Q \\
&\quad +\frac{4 c_{88,4} B_0\left(m_z+m_x\right)}{\alpha^{(3,\bar{3})}}\left\langle\pi^{+}\left|\Theta^{(3,\bar{3})}\right| K^{+}\right\rangle,
\end{aligned} \tag{31}
$$

where by CPS symmetry the power divergence is removed to all orders in χPT. Through NLO in χPT we have,

$$
\left\langle\pi^{+}\left|\mathcal{O}_{\mathrm{sub}}^{(8,8)(1/2)}\right| K^{+}\right\rangle_Q=\left\langle\pi^{+}\left|\mathcal{O}^{(8,8)(1/2)}\right| K^{+}\right\rangle_Q-\frac{8}{f^2} c_{88,4} m_{xz}^2 . \tag{32}
$$

These relations hold for either $Q=Q_1, Q_2$, and lead to

$$
\begin{aligned}
\left\langle\pi^{+}\left|\mathcal{O}_{\mathrm{sub}}^{(8,8)(1/2)}\right| K^{+}\right\rangle_{Q_1} &= \frac{8\alpha_{88}}{f^2}+\frac{4}{f^2}\Big\{\left(-c_{88,1}+c_{88,2}+2 c_{88,3}\right) m_{xz} m_X+4 c_{88,4}\left(m_{xz}^2+m_X^2\right) \\
&\quad +4 c_{88,5}\left(m_{xz}^2+m_X^2\right)+4 c_{88,6}\left(2 m_D^2+m_S^2\right)\Big\} \\
&\quad +\left\langle\pi^{+}\left|\mathcal{O}^{(8,8)(1/2)}\right| K^{+}\right\rangle_{Q_1,\mathrm{logs}}
\end{aligned} \tag{33}
$$

$$\left\langle \pi^+ \left| \mathcal{O}^{(8,8)(1/2)}_{\rm sub} \right| K^+ \right\rangle_{Q_2} = \left\langle \pi^+ \left| \mathcal{O}^{(8,8)(1/2)}_{\rm sub} \right| K^+ \right\rangle_{Q_1} + \frac{4}{3}\frac{\alpha_{88}}{f^4}\left\{ 3\frac{m_{xz}}{m_{xz}-m_X}\left[\ell(m_X^2)+\ell(m_Z^2)-2\ell(m_{xz}^2)\right] + 6m_{xz}m_X(\beta(q^2,m_{xz}^2,m_X^2)-\beta(q^2,m_{xz}^2,m_Z^2))\right\} . \tag{34}$$

The logarithms appearing in Eq. (33) are given in Appendix E.

When we have degenerate valence quark masses ($m_x = m_y = m_z$), the above formula can be simplified. However, for some terms, especially those which involve residue functions $R_X(m_a, m_b)$, taking this limit is non-trivial. Thus, we give the degenerate valence $K \to \pi$ amplitudes explicitly. (The degenerate valence $K \to 0$ matrix elements vanish due to CPS symmetry [34].) The subtracted amplitude for the degenerate case is given by

$$\begin{aligned}\left\langle \pi^+ \left| \mathcal{O}^{(8,8)(1/2)}_{\rm sub} \right| K^+ \right\rangle_Q^{\rm deg.val.} &= \left\langle \pi^+ \left| \mathcal{O}^{(8,8)(1/2)} \right| K^+ \right\rangle_Q^{\rm deg.val.} + \frac{4c_{88,4}(2B_0m_x)}{\alpha^{(3,\bar{3})}}\left\langle \pi^+ \left| \Theta^{(3,\bar{3})} \right| K^+ \right\rangle \\ &= \left\langle \pi^+ \left| \mathcal{O}^{(8,8)(1/2)} \right| K^+ \right\rangle_Q^{\rm deg.val.} - \frac{8}{f^2}c_{88,4}m_X^2 ,\end{aligned} \tag{35}$$

where again the second equality is correct through NLO in χPT. In the degenerate case the amplitudes are the same for Q_1 and Q_2,

$$\begin{aligned}\left\langle \pi^+ \left| \mathcal{O}^{(8,8)(3/2)} \right| K^+ \right\rangle_{Q_{1,2}}^{\rm deg.val.} =& \frac{4\alpha_{88}}{f^2}\left(1+\delta Z_X\right) \\ &+ \frac{4\alpha_{88}}{f^4}\left\{ -\frac{8}{3}\left[2\ell\left(m_{xd}^2\right)+\ell\left(m_{xs}^2\right)\right] + 2m_X^2\tilde{\ell}\left(m_X^2\right)\right\} \\ &+ \frac{4}{f^2}\left[\left(-c_{88,1}-c_{88,2}+4c_{88,4}+4c_{88,5}\right)m_X^2 \right. \\ &\left. + 2c_{88,6}\left(2m_D^2+m_S^2\right)\right],\end{aligned} \tag{36}$$

$$\begin{aligned}\left\langle \pi^+ \left| \mathcal{O}^{(8,8)(1/2)}_{sub} \right| K^+ \right\rangle_{Q_{1,2}}^{\rm deg.val.} =& \frac{8\alpha_{88}}{f^2}\left(1+\delta Z_X\right) \\ &+ \frac{8\alpha_{88}}{f^4}\left\{ -\frac{8}{3}\left[2\ell\left(m_{xd}^2\right)+\ell\left(m_{xs}^2\right)\right] - m_X^2\tilde{\ell}\left(m_X^2\right)\right. \\ &+ \frac{4}{f^2}\left[\left(-c_{88,1}+c_{88,2}+2c_{88,3}+8c_{88,4}+8c_{88,5}\right)m_X^2 \right. \\ &\left. + 4c_{88,6}\left(2m_D^2+m_S^2\right)\right].\end{aligned} \tag{37}$$

C. $K \to \pi\pi$ amplitudes in full QCD

The LEC's needed to construct the (8,8), $K \to \pi\pi$ amplitudes at physical kinematics through NLO can be obtained from the $K \to \pi$ and $K \to 0$ amplitudes given above. The

extraction of LEC's is essentially unchanged from the case of 3 degenerate sea quarks treated in Ref. [11]. For completeness, we present the physical $K \to \pi\pi$ amplitudes, which were calculated originally in Refs. [8, 9], and subsequently checked in Refs. [11, 35].

$$\langle \pi^+\pi^- | \mathcal{O}^{(8,8),(3/2)} | K^0 \rangle = -\frac{4i\alpha_{88}}{f_K f_\pi^2} + \frac{12i}{f_K f_\pi^2} \big[(-c_{88,2} - c_{88,3} - 2c_{88,4} - 2c_{88,5} - 4c_{88,6}) m_K^2 \\ - (-c_{88,1} - c_{88,2} + 4c_{88,4} + 4c_{88,5} + 2c_{88,6}) m_\pi^2 \big] + \langle \pi^+\pi^- | \mathcal{O}^{(8,8),(3/2)} | K^0 \rangle_{\text{logs}}, \tag{38}$$

$$\langle \pi^+\pi^- | \mathcal{O}^{(8,8),(1/2)} | K^0 \rangle = -\frac{8i\alpha_{88}}{f_K f_\pi^2} - \frac{12i}{f_K f_\pi^2} \big[(-c_{88,1} - c_{88,2} + 4c_{88,4} + 4c_{88,5} + 8c_{88,6}) m_K^2 \\ + (-c_{88,1} + c_{88,2} + 2c_{88,3} + 8c_{88,4} + 8c_{88,5} + 4c_{88,6}) m_\pi^2 \big] + \langle \pi^+\pi^- | \mathcal{O}^{(8,8),(1/2)} | K^0 \rangle_{\text{logs}}. \tag{39}$$

The logarithmic terms are given in Appendix E. Note that in Eqs. (38) and (39), the decay constants appearing in the tree-level terms are the physical decay constants (correct to one loop). When constructing $K \to \pi\pi$ amplitudes using Eqs. (38) and (39), one should use the physical decay constants in the tree-level expression, as determined from lattice calculations or experiment, in order to avoid double counting a subset of the one-loop corrections.

VI. $K \to \pi$ FOR THE $(27,1)$, $\Delta I = 3/2$ CASE

The operators which transform as (27,1)'s under the irreducible representation of the chiral symmetry group give the dominant contribution to $\mathrm{Re}(A_2)$, *i.e.*, the real part of the $\Delta I = 3/2$, $K \to \pi\pi$ amplitude. In this section we review the (27,1) chiral operators that are needed through NLO, and we give results for the NLO $\Delta I = 3/2$, $K \to \pi$ amplitude.

A. Definition of the $\mathcal{O}^{(27,1)}$ operators

Following [7, 11], the operator in the (27,1) representation can be written as

$$\begin{aligned}
\mathcal{O}^{(27,1)} =& T^{ij}_{kl} \left(\Sigma\partial_\mu\Sigma^\dagger\right)^k_i \left(\Sigma\partial^\mu\Sigma^\dagger\right)^l_j \\
&+ c_{27,1} T^{ij}_{kl} \left(S\right)^k_i \left(S\right)^l_j \\
&+ c_{27,2} T^{ij}_{kl} \left(P\right)^k_i \left(P\right)^l_j \\
&+ c_{27,4} T^{ij}_{kl} \left(L_\mu\right)^k_i \left(\{L^\mu, S\}\right)^l_j \\
&+ c_{27,5} T^{ij}_{kl} \left(L_\mu\right)^k_i \left([L^\mu, P]\right)^l_j \\
&+ c_{27,6} T^{ij}_{kl} \left(S\right)^k_i \left(L^2\right)^l_j \\
&+ c_{27,7} T^{ij}_{kl} \left(L_\mu\right)^k_i \left(L^\mu\right)^l_j \,\mathrm{str}\left[S\right] \\
&+ c_{27,20} T^{ij}_{kl} \left(L_\mu\right)^k_i \left(\partial_\nu W^{\mu\nu}\right)^l_j \\
&+ c_{27,24} T^{ij}_{kl} \left(W_{\mu\nu}\right)^k_i \left(W^{\mu\nu}\right)^l_j
\end{aligned} \tag{40}$$

where S, P, L_μ are defined in Eq. (11) and

$$W_{\mu\nu} = 2\left(\partial_\mu L_\nu + \partial_\nu L_\mu\right). \tag{41}$$

The tensor T^{ij}_{kl} has different elements depending on which isospin we are projecting. To project the $\Delta I = 3/2$ operator, we set

$$\begin{aligned}
T^{13}_{12} = T^{31}_{12} = T^{13}_{21} = T^{31}_{21} &= \frac{1}{2} \\
T^{23}_{22} = T^{32}_{22} &= -\frac{1}{2} ,
\end{aligned} \tag{42}$$

whereas for the $\Delta I = 1/2$ operator, we set

$$\begin{aligned}
T^{13}_{12} = T^{31}_{12} = T^{13}_{21} = T^{31}_{21} &= \frac{1}{2} \\
T^{23}_{22} = T^{32}_{22} &= 1 \\
T^{33}_{32} = T^{33}_{23} &= -\frac{3}{2} .
\end{aligned} \tag{43}$$

In order to adapt Eq. (40) to the partially quenched theory, we must promote T to a 9^4 element tensor, although many components will remain zero (only the 3^4 block corresponding to the valence quark sector will have non-zero elements). To take into account the graded nature of the group, we multiply by factors of ε_i defined in Eq. (15), such that

$$\mathcal{O}^{(27,1)} = \sum_{ijkl} \varepsilon_i \varepsilon_j T^{ij}_{kl} \left(\Sigma\partial_\mu\Sigma^\dagger\right)^k_i \left(\Sigma\partial^\mu\Sigma^\dagger\right)^l_j \ , \tag{44}$$

where we display the summation over i, j, k, l explicitly for clarity.

There is another equivalent approach to obtaining the partially quenched operators for the (27,1) case. It is possible, as illustrated in Appendix D of [28], to rewrite Eq. (40) in terms of traces over the various operators. The partially quenched theory is then obtained in the usual way by changing traces to supertraces, and we obtain

$$O^{(27,1),(3/2)} = \mathrm{str}[\lambda_6 \Sigma \partial_\mu \Sigma^\dagger] \mathrm{str}[A \Sigma \partial^\mu \Sigma^\dagger] + \mathrm{str}[\lambda_3 \Sigma \partial_\mu \Sigma^\dagger] \mathrm{str}[\lambda_4 \Sigma \partial^\mu \Sigma^\dagger], \tag{45}$$

$$O^{(27,1),(1/2)} = \mathrm{str}[\lambda_6 \Sigma \partial_\mu \Sigma^\dagger] \mathrm{str}[B \Sigma \partial^\mu \Sigma^\dagger] + \mathrm{str}[\lambda_3 \Sigma \partial_\mu \Sigma^\dagger] \mathrm{str}[\lambda_4 \Sigma \partial^\mu \Sigma^\dagger] \ , \tag{46}$$

where we have defined the matrices

$$(\lambda_3)_{ij} = \delta_{i3}\delta_{j1}, \qquad (\lambda_4)_{ij} = \delta_{i1}\delta_{j2}, \tag{47}$$

$$A_{ij} = \delta_{i1}\delta_{j1} - \delta_{i2}\delta_{j2}, \qquad B = \mathrm{diag}(1, 2, -3). \tag{48}$$

Since the kaon has isospin $I = 1/2$, the $\Delta I = 3/2, K \to 0$ process vanishes. The amplitude for the (27,1), $\Delta I = 3/2$, $K \to \pi$ matrix element is

$$\begin{aligned} \left\langle \pi^+ \left| \mathcal{O}^{(27,1)(3/2)} \right| K^+ \right\rangle = & -\frac{4\alpha_{27}}{f^2} m_X m_{xz} + \frac{1}{f^2} \Big[16 \left(-c_{27,2} + 4c_{27,24} \right) m_X^2 m_{xz}^2 \\ & + 8 \left(c_{27,4} - c_{27,20} \right) m_X m_{xz} \left(m_X^2 + m_{xz}^2 \right) \\ & + 8c_{27,7} m_X m_{xz} \left(2m_D^2 + m_S^2 \right) \Big] + \left\langle \pi^+ \left| \mathcal{O}^{(27,1)(3/2)} \right| K^+ \right\rangle_{\mathrm{logs}}, \end{aligned} \tag{49}$$

where the logarithmic terms are given in Appendix F.

For degenerate valence quarks, this amplitude simplifies to

$$\begin{aligned} \left\langle \pi^+ \left| \mathcal{O}^{(27,1)(3/2)} \right| K^+ \right\rangle^{\mathrm{deg.val.}} = & -\frac{4\alpha_{27}}{f^2} m_X^2 \left[1 + \delta Z_X + \frac{\delta m_X^2}{m_X^2} \right] \\ & + \frac{8}{3} \frac{\alpha_{27}}{f^4} m_X^2 \left\{ 6\ell(m_X^2) + 8\ell(m_{xd}^2) + 4\ell(m_{xs}^2) - 3m_X^2 \tilde{\ell}(m_X^2) \right\} \\ & + \frac{1}{f^2} \Big[16 \left(-c_{27,2} + c_{27,4} - c_{27,20} + 4c_{27,24} \right) m_X^4 \\ & + 8c_{27,7} m_X^2 \left(2m_D^2 + m_S^2 \right) \Big] \ . \end{aligned} \tag{50}$$

VII. $\Delta I = 1/2$ WEAK MATRIX ELEMENTS FOR (8,1) AND (8,1)+(27,1) OPERATORS

In this section we present results for $\Delta I = 1/2$ amplitudes, which include $K \to 0$ and $K \to \pi$ matrix elements, for operators that transform under the $(8, 1) + (27, 1)$ representation, and for those that transform under the pure (8,1) representation. We perform the

power subtraction explicitly through NLO in the chiral expansion and present the subtracted amplitudes.

A. Definition of the $\mathcal{O}^{(8,1)}$ operator

As shown in Ref. [7], there are two leading order $[\mathcal{O}(p^2)]$ operators in the chiral symmetry group (8,1), with coefficients α_1 and α_2. There are 13 NLO $[\mathcal{O}(p^4)]$ operators relevant for this work. Note that there is an extra operator which only appears in the partially quenched case. In the full theory, by convention [36], operator 14 is absorbed into operators 10, 11, 12, and 13 via the Cayley-Hamilton theorem. This is not possible in the partially quenched theory [11, 27]. The (8,1) operators are

$$\begin{aligned}\mathcal{O}^{(8,1)} =&\alpha_1 \mathrm{str}\left[\lambda_6 \partial_\mu \Sigma \partial^\mu \Sigma^\dagger\right] + \alpha_2 2B_0 \mathrm{str}\left[\lambda_6\left(\mathcal{M}^\dagger\Sigma^\dagger + \Sigma\mathcal{M}\right)\right] \\ &+ c_{81,1}\mathrm{str}\left[\lambda_6 S^2\right] + c_{81,2}\mathrm{str}\left[\lambda_6 S\right]\mathrm{str}\left[S\right] \\ &+ c_{81,3}\mathrm{str}\left[\lambda_6 P^2\right] + c_{81,4}\mathrm{str}\left[\lambda_6 P\right]\mathrm{str}\left[P\right] \\ &+ c_{81,5}\mathrm{str}\left[\lambda_6\left[P,S\right]\right] + c_{81,10}\mathrm{str}\left[\lambda_6\left\{S,L^2\right\}\right] \\ &+ c_{81,11}\mathrm{str}\left[\lambda_6 L_\mu S L^\mu\right] + c_{81,12}\mathrm{str}\left[\lambda_6 L_\mu\right]\mathrm{str}\left[\left\{L^\mu,S\right\}\right] \\ &+ c_{81,13}\mathrm{str}\left[\lambda_6 S\right]\mathrm{str}\left[L^2\right] + c_{81,14}\mathrm{str}\left[\lambda_6 L^2\right]\mathrm{str}\left[S\right] \\ &+ c_{81,15}\mathrm{str}\left[\lambda_6\left[L^2,P\right]\right] + c_{81,35}\mathrm{str}\left[\lambda_6\left\{L_\mu,\partial_\nu W^{\mu\nu}\right\}\right] \\ &+ c_{81,39}\mathrm{str}\left[\lambda_6 W_{\mu\nu}W^{\mu\nu}\right]\end{aligned} \tag{51}$$

The (27,1) operators relevant for this section are given in Eq. (40).

B. $K \to 0$ amplitudes

The NLO expression for the unsubtracted $K \to 0$ amplitude in the pure (8,1) case is

$$\begin{aligned}\left\langle 0\left|\mathcal{O}^{(8,1)}\right|K^0\right\rangle =& \frac{4i}{f}\alpha_2(m_{xz}^2 - m_X^2) + \frac{8i}{f}\left(m_{xz}^2 - m_X^2\right)\left[2\left(c_{81,1} - c_{81,5}\right)m_{xz}^2\right. \\ &\left. + c_{81,2}\left(2m_D^2 + m_S^2\right)\right] + \left\langle 0\left|\mathcal{O}^{(8,1)}\right|K^0\right\rangle_{\mathrm{logs}},\end{aligned} \tag{52}$$

where the logarithmic terms are given in Appendix G. For the (8,1)+(27,1) case, we have

$$\begin{aligned}\left\langle 0\left|\mathcal{O}^{(8,1)+(27,1)(1/2)}\right|K^0\right\rangle =& \left\langle 0\left|\mathcal{O}^{(27,1)}\right|K^0\right\rangle_{\mathrm{logs}} \\ &+ \frac{48i}{f}c_{27,1}\left(m_{xz}^2 - m_X^2\right)^2 + \left\langle 0\left|\mathcal{O}^{(8,1)}\right|K^0\right\rangle,\end{aligned} \tag{53}$$

where

$$
\begin{aligned}
\left\langle 0 \left| \mathcal{O}^{(27,1)} \right| K^0 \right\rangle_{\text{logs}} = \ & \frac{4i}{f^3}\alpha_{27}\Bigg\{ \Bigg[2m_X^2 - R_X(m_\eta) + m_X^2 R_\eta(m_X, m_X) \\
& + 2m_X^2 R_X(m_\eta, m_Z) \Bigg] \ell(m_X^2) + \Bigg[2m_Z^2 - R_Z(m_\eta) \\
& + m_Z^2 R_\eta(m_Z, m_Z) + 2m_Z^2 R_Z(m_\eta, m_X) \Bigg] \ell(m_Z^2) \\
& - 6m_{xz}^2 \ell(m_{xz}^2) + \Bigg[-m_\eta^2 R_\eta(m_Z, m_Z) + 2m_\eta^2 R_\eta(m_X, m_Z) \\
& - m_\eta^2 R_\eta(m_X, m_X) \Bigg] \ell(m_\eta^2) + m_X^2 R_X(m_\eta) \tilde{\ell}(m_X^2) \\
& + m_Z^2 R_Z(m_\eta) \tilde{\ell}(m_Z^2) \Bigg\}.
\end{aligned}
\tag{54}
$$

Following the procedure given in Ref. [28], we perform the subtraction of the power divergence in (8,1) and (8,1)+(27,1) amplitudes. In order to do this we require the amplitudes for $\Theta^{(3,\bar{3})}$ through NLO, given in Section IV. The ratio of (8,1) and $(3,\overline{3})$, $K \to 0$ matrix elements to NLO is

$$
\begin{aligned}
\frac{\left\langle 0 \left| \mathcal{O}^{(8,1)} \right| K^0 \right\rangle}{\left\langle 0 \left| \Theta^{(3,\bar{3})} \right| K^0 \right\rangle} = & 2\frac{\alpha_2}{\alpha^{(3,\bar{3})}} B_0 \left(m_z - m_x \right) + \frac{f}{2i\alpha^{(3,\bar{3})}} \langle 0 | \mathcal{O}^{(8,1)} | K^0 \rangle_{\text{logs}}^{(1)} \\
& + \frac{4}{\alpha^{(3,\bar{3})}} \left(m_{xz}^2 - m_X^2 \right) \left[2 \left(c'_{81,1} - c'_{81,5} \right) m_{xz}^2 + c'_{81,2} \left(2m_D^2 + m_S^2 \right) \right],
\end{aligned}
\tag{55}
$$

where the transformed coefficients $c'_{81,1}$, $c'_{81,2}$, $c'_{81,5}$ [defined in Table I] are linear combinations of the original LEC's, $c_{81,1}$, etc., and the Gasser-Leutwyler coefficients originating from $\mathcal{O}(p^2)$ terms in the amplitude $\left\langle 0 \left| \Theta^{(3,\bar{3})} \right| K^0 \right\rangle$. The first term on the right-hand side of Eq. (55) contains the power divergence, which is proportional to $m_z - m_x$ to all orders in the chiral expansion. The remaining terms are finite, including the rotated LEC's $c'_{81,i}$. Since the rotated LEC's contain a term proportional to α_2, it follows that the unrotated $c_{81,i}$'s must also contain power divergences [37]. This was implicitly assumed in the work of Ref. [11].

Table I: The transformation of the (8,1) LEC's (denoted in the text with a prime) under the vacuum subtraction process.

LEC	Transformed LEC
$c_{81,1}$	$c_{81,1} - (4\alpha_2/f^2)(2L_8 + H_2)$
$c_{81,2}$	$c_{81,2} - (16\alpha_2/f^2)L_6$
$c_{81,3}$	$c_{81,3} + (4\alpha_2/f^2)(-2L_8 + H_2)$
$c_{81,5}$	$c_{81,5} - (4\alpha_2/f^2)H_2$
$c_{81,10}$	$c_{81,10} - (4\alpha_2/f^2)L_5$
$c_{81,13}$	$c_{81,13} - (8\alpha_2/f^2)L_4$
$c_{81,15}$	$c_{81,15} + (4\alpha_2/f^2)L_5$

A similar expression exists for the ratio involving the (8,1)+(27,1) amplitude,

$$\begin{aligned}\frac{\left\langle 0\left|\mathcal{O}^{(8,1)+(27,1)(1/2)}\right|K^0\right\rangle}{\left\langle 0\left|\Theta^{(3,\bar{3})}\right|K^0\right\rangle} =& 2\frac{\alpha_2}{\alpha^{(3,\bar{3})}}B_0\left(m_z - m_x\right) + \frac{f}{2i\alpha^{(3,\bar{3})}}\langle 0|\mathcal{O}^{(8,1)}|K^0\rangle^{(1)}_{\text{logs}} \\ &+ \frac{f}{2i\alpha^{(3,\bar{3})}}\langle 0|\mathcal{O}^{(27,1)}|K^0\rangle_{\text{logs}} \\ &+ \frac{4}{\alpha^{(3,\bar{3})}}\left(m_{xz}^2 - m_X^2\right)\left[2\left(c'_{81,1} - c'_{81,5}\right)m_{xz}^2 + c'_{81,2}\left(2m_D^2 + m_S^2\right)\right. \\ &\left. + 6c_{27,1}(m_{xz}^2 - m_X^2)\right]. \end{aligned} \tag{56}$$

C. $K \to \pi$ amplitudes with 2+1 valence flavors

For the $K \to \pi$ amplitudes, we first present the matrix elements of the unsubtracted operators,

$$\begin{aligned}\left\langle \pi^+\left|\mathcal{O}^{(8,1)}\right|K^+\right\rangle =& \frac{4\alpha_1}{f^2}m_{xz}m_X - \frac{4\alpha_2}{f^2}m_{xz}^2 \\ &+ \frac{8}{f^2}\left[-2c_{81,1}m_{xz}^4 - c_{81,2}m_{xz}^2\left(2m_D^2 + m_S^2\right) - 2c_{81,3}m_X^2m_{xz}^2\right. \\ &+ 2c_{81,5}m_{xz}^2\left(m_{xz}^2 - m_X^2\right) + 2c_{81,10}m_Xm_{xz}^3 + c_{81,11}m_X^3m_{xz} \\ &+ c_{81,14}m_Xm_{xz}\left(2m_D^2 + m_S^2\right) - 2c_{81,35}m_Xm_{xz}\left(m_X^2 + m_{xz}^2\right) \\ &\left. + 8c_{81,39}m_X^2m_{xz}^2\right] + \left\langle \pi^+\left|\mathcal{O}^{(8,1)}\right|K^+\right\rangle_{\text{logs}}, \end{aligned} \tag{57}$$

$$
\begin{aligned}
\left\langle\pi^+\left|\mathcal{O}^{(8,1)+(27,1)(1/2)}\right|K^+\right\rangle \;=\; & -\frac{4\alpha_{27}}{f^2}m_{xz}m_X+\frac{1}{f^2}\Big[48c_{27,1}m_{xz}^2\left(m_{xz}^2-m_X^2\right)\\
& +16\left(-c_{27,2}+4c_{27,24}\right)m_X^2m_{xz}^2\\
& +8\left(c_{27,4}-c_{27,20}\right)m_Xm_{xz}\left(m_X^2+m_{xz}^2\right)\\
& +24c_{27,6}m_Xm_{xz}\left(m_X^2-m_{xz}^2\right)+8c_{27,7}m_Xm_{xz}\left(2m_D^2+m_S^2\right)\Big]\\
& +\left\langle\pi^+\left|\mathcal{O}^{(27,1)(1/2)}\right|K^+\right\rangle_{\rm logs}+\left\langle\pi^+\left|\mathcal{O}^{(8,1)}\right|K^+\right\rangle, \qquad (58)
\end{aligned}
$$

where the logarithmic contributions to the (8,1) amplitude are given in Appendix G, and the logarithmic contributions to the (27,1) amplitude are given in Appendix F.

Given the coefficient of the power divergent term from $K\to 0$, it is possible to carry out the operator subtraction in $K\to\pi$ numerically. By CPS symmetry, the following subtraction removes the divergence to all orders in the chiral expansion,

$$
\left\langle\pi^+\left|\mathcal{O}^{(8,1)}_{\rm sub}\right|K^+\right\rangle \;=\; \left\langle\pi^+\left|\mathcal{O}^{(8,1)}\right|K^+\right\rangle-2\frac{\alpha_2}{\alpha^{(3,\bar{3})}}B_0(m_z+m_x)\left\langle\pi^+\left|\Theta^{(3,\bar{3})}\right|K^+\right\rangle. \qquad (59)
$$

To NLO this expression becomes

$$
\begin{aligned}
\left\langle\pi^+\left|\mathcal{O}^{(8,1)}_{\rm sub}\right|K^+\right\rangle \;=\; & \frac{4\alpha_1}{f^2}m_{xz}m_X+\frac{8}{f^2}\Big[-2c'_{81,1}m_{xz}^4-c'_{81,2}m_{xz}^2\left(2m_D^2+m_S^2\right)\\
& -2c'_{81,3}m_X^2m_{xz}^2+2c'_{81,5}m_{xz}^2\left(m_{xz}^2-m_X^2\right)+2c'_{81,10}m_Xm_{xz}^3\\
& +c_{81,11}m_X^3m_{xz}+c_{81,14}m_Xm_{xz}\left(2m_D^2+m_S^2\right)\\
& -2c_{81,35}m_Xm_{xz}\left(m_X^2+m_{xz}^2\right)+8c_{81,39}m_X^2m_{xz}^2\Big]\\
& +\left\langle\pi^+\left|\mathcal{O}^{(8,1)}\right|K^+\right\rangle^{(1)}_{\rm logs}, \qquad (60)
\end{aligned}
$$

where again, the $c'_{81,i}$ are the linear combinations of LEC's given in Table I. Note that the subtraction eliminates the term in Eq. (57) proportional to α_2. The NLO chiral logarithms proportional to α_2 are also eliminated. The remaining logarithms are contained in the term $\left\langle\pi^+\left|\mathcal{O}^{(8,1)}\right|K^+\right\rangle^{(1)}_{logs}$ given in Appendix G; this term is proportional to α_1.

A similar subtraction can be performed for the (8,1)+(27,1) case,

$$
\begin{aligned}
\left\langle\pi^+\left|\mathcal{O}^{(8,1)+(27,1)(1/2)}_{\rm sub}\right|K^+\right\rangle \;=\; & \left\langle\pi^+\left|\mathcal{O}^{(8,1)+(27,1)(1/2)}\right|K^+\right\rangle\\
& -2\frac{\alpha_2}{\alpha^{(3,\bar{3})}}B_0(m_z+m_x)\left\langle\pi^+\left|\Theta^{(3,\bar{3})}\right|K^+\right\rangle. \qquad (61)
\end{aligned}
$$

Again, this subtraction removes the power divergences to all orders in the chiral expansion. To NLO the subtracted operator gives the matrix element,

$$
\begin{aligned}
\left\langle \pi^+ \left| \mathcal{O}_{\rm sub}^{(8,1)+(27,1)(1/2)} \right| K^+ \right\rangle = & -\frac{4\alpha_{27}}{f^2} m_{xz} m_X + \frac{1}{f^2}\Big[48 c_{27,1} m_{xz}^2 \left(m_{xz}^2 - m_X^2\right) \\
& + 16\left(-c_{27,2} + 4c_{27,24}\right) m_X^2 m_{xz}^2 \\
& + 8\left(c_{27,4} - c_{27,20}\right) m_X m_{xz} \left(m_X^2 + m_{xz}^2\right) \\
& + 24 c_{27,6} m_X m_{xz} \left(m_X^2 - m_{xz}^2\right) + 8 c_{27,7} m_X m_{xz} \left(2m_D^2 + m_S^2\right)\Big] \\
& + \left\langle \pi^+ \left| \mathcal{O}^{(27,1)(1/2)} \right| K^+ \right\rangle_{\rm logs} + \left\langle \pi^+ \left| \mathcal{O}_{\rm sub}^{(8,1)} \right| K^+ \right\rangle . \qquad (62)
\end{aligned}
$$

For degenerate valence quarks, Eq. (60) reduces to

$$
\begin{aligned}
\left\langle \pi^+ \left| \mathcal{O}_{\rm sub}^{(8,1)} \right| K^+ \right\rangle^{\rm deg.val.} = & \frac{4}{f^2} m_X^2 \Big\{ \alpha_1 + 2\Big[\big(-2c'_{81,1} - 2c'_{81,3} + 2c'_{81,10} + c_{81,11} \\
& - 4c_{81,35} + 8c_{81,39} \big) m_X^2 + \left(-c'_{81,2} + c_{81,14}\right)\left(2m_D^2 + m_S^2\right)\Big]\Big\} \\
& + \left\langle \pi^+ \left| \mathcal{O}^{(8,1)} \right| K^+ \right\rangle_{logs}^{\rm deg.val.,(1)} , \qquad (63)
\end{aligned}
$$

where the logarithmic terms are given in Appendix G. For the $(8,1)+(27,1)$, $K \to \pi$ matrix element, we have for the degenerate valence case,

$$
\begin{aligned}
\left\langle \pi^+ \left| \mathcal{O}_{\rm sub}^{(8,1)+(27,1)(1/2)} \right| K^+ \right\rangle^{\rm deg.val.} = & -\frac{4\alpha_{27}}{f^2} m_X^2 \left[1 + \delta Z_X + \frac{\delta m_X^2}{m_X^2} \right] \\
& + \frac{8}{3}\frac{\alpha_{27}}{f^4} m_X^2 \Big\{ 6\ell(m_X^2) + 8\ell(m_{xd}^2) \\
& + 4\ell(m_{xs}^2) - 3m_X^2 \tilde{\ell}(m_X^2) \Big\} \\
& + \frac{1}{f^2}\Big[16\left(-c_{27,2} + c_{27,4} - c_{27,20} + 4c_{27,24}\right) m_X^4 \\
& + 8c_{27,7} m_X^2 \left(2m_D^2 + m_S^2\right)\Big] \\
& + \left\langle \pi^+ \left| \mathcal{O}_{sub}^{(8,1)} \right| K^+ \right\rangle^{\rm deg.val.} . \qquad (64)
\end{aligned}
$$

Note that for degenerate valence quarks the (27,1), $\Delta I = 1/2$ amplitude is the same as the $(27,1), \Delta I = 3/2$ amplitude, Eq. (50).

VIII. FINITE VOLUME CORRECTIONS

Incorporating the leading corrections coming from the finite volume used in lattice simulations for the above expressions is straightforward. Here we assume that the time extent used to extract the above matrix elements is infinite, and that the only corrections come from the finite spatial volume. There are two classes of one-loop integrals that must be replaced by their finite volume counterparts. The first is defined in Eq. (B1), and its associated double-pole counterparts are defined in Eqs. (B2) and (B3) [these are related to Eq. (B1) by derivatives with respect to m^2]. As discussed in Refs. [38, 39], finite volume effects can be accounted for by making the replacements

$$\ell(m^2) \to \ell(m^2) + \frac{1}{16\pi^2} m^2 \delta_1(mL) \ , \tag{65}$$

$$\tilde{\ell}(m^2) \to \tilde{\ell}(m^2) + \frac{1}{16\pi^2} \delta_3(mL) \ , \tag{66}$$

$$\tilde{\tilde{\ell}}(m^2) \to \tilde{\tilde{\ell}}(m^2) + \frac{1}{16\pi^2} \frac{\delta_5(mL)}{m^2} \ , \tag{67}$$

with

$$\delta_1(mL) = 4 \sum_{\mathbf{n}\neq 0} \frac{K_1(|\mathbf{n}|mL)}{|\mathbf{n}|mL} \ , \tag{68}$$

$$\delta_3(mL) = -\frac{\partial}{\partial m^2}\left[m^2 \delta_1(mL)\right] = 2\sum_{\mathbf{n}\neq 0} K_0(|\mathbf{n}|mL) \ , \tag{69}$$

$$\delta_5(mL) = m^2 \frac{\partial}{\partial m^2}\left[\delta_3(mL)\right] = -\sum_{\mathbf{n}\neq 0} (|\mathbf{n}|mL) K_1(|\mathbf{n}|mL) \ , \tag{70}$$

with K_0, K_1 the modified Bessel functions of imaginary argument.

The second class of loop integrals are more complicated, and are defined in Eqs. (B4) and (B5). For these, we recall the technique used to calculate the above finite volume corrections. We begin with the finite volume Euclidean space version of Eq. (B4), and apply the Poisson Resummation Formula (as discussed in Refs. [38, 40]). This leads to the following replacements,

$$\beta\left(q^2, m_1^2, m_2^2\right) \to \beta\left(q^2, m_1^2, m_2^2\right) + \frac{1}{4\pi^2}\delta_\beta(qL, m_1 L, m_2 L) \ , \tag{71}$$

$$\tilde{\beta}\left(q^2, m_1^2, m_2^2\right) \ \to \ \tilde{\beta}\left(q^2, m_1^2, m_2^2\right) - \frac{1}{4\pi^2 m_1^2}\delta_{\tilde{\beta}}(qL, m_1 L, m_2 L), \tag{72}$$

and the corrections

$$\delta_\beta(qL, m_1L, m_2L) \equiv \sum_{\mathbf{n}\neq 0} \int_0^\infty dk \frac{k\sin(k|\mathbf{n}|)}{|\mathbf{n}|} \frac{\omega_1+\omega_2}{\omega_1\omega_2[(qL)^2+(\omega_1+\omega_2)^2]} \ , \tag{73}$$

$$\delta_{\tilde\beta}(qL, m_1L, m_2L) \equiv (m_1L)^2 \sum_{\mathbf{n}\neq 0} \int_0^\infty dk \frac{k\sin(k|\mathbf{n}|)}{|\mathbf{n}|} \frac{(qL)^2\omega_2+(2\omega_1+\omega_2)(\omega_1+\omega_2)^2}{2\omega_1^3\omega_2[(qL)^2+(\omega_1+\omega_2)^2]^2} \ , \tag{74}$$

where we have defined

$$\omega_i = \sqrt{k^2+(m_iL)^2} \ ,$$

and where the function in Eq. (74) is obtained by taking the partial derivative of Eq. (73) with respect to m_1^2.

These formulas can be simplified as in Ref. [40], but only in special cases (such as degenerate masses). For the general case, it is more difficult to find an approximate expression for these finite volume corrections.[2] However, it is relatively simple to evaluate these expressions numerically at a finite number of points. Given a set of lattice data at a number of quark masses and lattice volumes, it would be straightforward to tabulate the appropriate finite volume corrections from the above formulas.

IX. CONCLUSIONS

This paper presents the calculation of $K \to 0$ and $K \to \pi$ amplitudes to NLO in PQχPT with 2+1 flavors of non-degenerate sea quarks. Results are presented for both the $\Delta I = 1/2$ and 3/2 channels, for chiral operators corresponding to current-current, gluonic penguin, and electroweak penguin 4-quark operators. The chiral operators are conveniently grouped by their chiral transformation properties; this work computes matrix elements of (8,8), (27,1), (8,1), and (8,1)+(27,1) chiral operators. The power divergent operator subtraction is performed explicitly through NLO in the chiral expansion for $\Delta I = 1/2$ matrix elements. We have also shown how to include finite volume effects through one-loop for the quantities considered in this work. These results are useful for studying the chiral behavior of currently available 2+1 flavor lattice QCD results [41], from which the low energy constants of the chiral effective theory can be determined. The low energy constants of these matrix

[2] One cannot apply the expansion in Ref. [40], for example, because these integrals have three different relevant scales, as given by q^2, m_1^2, and m_2^2.

elements are necessary for an understanding of the $\Delta I = 1/2$ rule and for calculations of ϵ'/ϵ using current lattice QCD simulations. Electroweak penguin $K \to \pi\pi$ matrix elements can be constructed to NLO in χPT using the formulas presented in this work, allowing the convergence of the chiral expansion to be studied. This will serve as a useful cross-check for other non-χPT methods such as those proposed in Refs. [5, 42].

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Appendix A: ISOSPIN DECOMPOSITION

The operator that governs the transition $K \to \pi$ can have either isospin 1/2 or isospin 3/2. In typical lattice calculations, these two processes are calculated independently [28, 43], since the operator with isospin 1/2 mixes with a divergent lower dimensional operator, which must be subtracted. The isospin 3/2 amplitude does not have this complication. Therefore, we calculate the amplitudes for these two processes separately, making use of the amplitudes for $K^+ \to \pi^+$ and $K^0 \to \pi^0$.

We define $\mathcal{M}_+ = \langle \pi^+ |\mathcal{O}_i| K^+\rangle$, where $\mathcal{O}_i$ represents some $\Delta S = 1$ operator with both isospin 1/2 and 3/2 components, and $\mathcal{M}_0 = \langle \pi^0 |\mathcal{O}_i| K^0\rangle$. If we decompose the operator $\mathcal{O}_i$ by isospin, $\mathcal{O}_i = \mathcal{O}_i^{(3/2)} + \mathcal{O}_i^{(1/2)}$, then we have for the matrix elements,

$$\begin{aligned}
\mathcal{M}_+ &= \left\langle \pi^+ \left| \mathcal{O}_i^{(3/2)} \right| K^+ \right\rangle + \left\langle \pi^+ \left| \mathcal{O}_i^{(1/2)} \right| K^+ \right\rangle , \\
\mathcal{M}_0 &= \left\langle \pi^0 \left| \mathcal{O}_i^{(3/2)} \right| K^0 \right\rangle + \left\langle \pi^0 \left| \mathcal{O}_i^{(1/2)} \right| K^0 \right\rangle .
\end{aligned}$$

Given the relevant Clebsch-Gordon coefficients,

$$\frac{\left\langle \pi^+ \left| \mathcal{O}_i^{(3/2)} \right| K^+ \right\rangle}{\left\langle \pi^0 \left| \mathcal{O}_i^{(3/2)} \right| K^0 \right\rangle} = \frac{\sqrt{2}}{2}, \tag{A1}$$

$$\frac{\left\langle \pi^+ \left| \mathcal{O}_i^{(1/2)} \right| K^+ \right\rangle}{\left\langle \pi^0 \left| \mathcal{O}_i^{(1/2)} \right| K^0 \right\rangle} = -\sqrt{2}, \tag{A2}$$

we obtain the result,

$$\begin{aligned} \left\langle \pi^+ \left| \mathcal{O}_i^{(3/2)} \right| K^+ \right\rangle &= \frac{1}{3} \left(\mathcal{M}_+ + \sqrt{2}\mathcal{M}_0 \right) \\ \left\langle \pi^+ \left| \mathcal{O}_i^{(1/2)} \right| K^+ \right\rangle &= \frac{1}{3} \left(2\mathcal{M}_+ - \sqrt{2}\mathcal{M}_0 \right) \ . \end{aligned} \tag{A3}$$

Appendix B: LOOP FUNCTIONS AND RESIDUES

The following loop functions are used throughout this work, and they are regulated using dimensional regularization in the modified $\overline{\mathrm{MS}}$ scheme. For single-pole mesonic loops, we need

$$\ell\left(m^2\right) = \left[\lim_{d\to 4} \int \frac{d^d p}{(2\pi)^d} \frac{i}{p^2 - m^2 + i\epsilon}\right]_{\mathrm{reg}} = \frac{1}{16\pi^2} m^2 \ln\left(\frac{m^2}{\mu^2}\right) \ , \tag{B1}$$

(cf. $f^2 A\left(m^2\right)$ in Ref. [10, 11]). We also need the double pole expression

$$\begin{aligned} \tilde{\ell}\left(m^2\right) &= -\frac{\partial}{\partial m^2} \ell\left(m^2\right) \\ &= -\int \frac{d^d p}{(2\pi)^d} \frac{i}{\left(p^2 - m^2\right)^2} \ , \end{aligned} \tag{B2}$$

where the minus sign is chosen to be consistent with the form of Euclidean $\tilde{\ell}\left(m^2\right)$ in Refs. [39, 44].[3] Further, we will sometimes need

$$\tilde{\tilde{\ell}}\left(m^2\right) = \frac{\partial}{\partial m^2} \tilde{\ell}\left(m^2\right) . \tag{B3}$$

The two-point loop-function encountered in loops with strong-weak vertices and only a

[3] Note, however, that our definitions of ℓ and $\tilde{\ell}$ differ from Refs. [39, 44] by a factor of $1/16\pi^2$.

single pole is defined as:

$$\beta\left(q^2,m_1^2,m_2^2\right) = \left[i\int\frac{d^dp}{(2\pi)^d}\frac{1}{\left(p^2-m_1^2\right)\left((p+q)^2-m_2^2\right)}\right]_{\rm reg}$$
$$= \frac{1}{(4\pi)^2}\int_0^1 dx\left\{1+\ln\left[-x(1-x)q^2+(1-x)m_1^2+xm_2^2\right]-\ln\left(\mu^2\right)\right\}. \tag{B4}$$

Note that we always have $q^2 = (m_{xz}-m_X)^2$ for $K\to\pi$ amplitudes. This function is proportional to the B_0 function defined in Eq (A2) of [10]. Similar loops with double-poles require

$$\tilde{\beta}\left(q^2,m_1^2,m_2^2\right) = \frac{\partial}{\partial\left(m_1^2\right)}\beta\left(q^2,m_1^2,m_2^2\right) \ . \tag{B5}$$

To simplify the expressions, we use the notation for the residues arising from disconnected meson propagators,

$$R_x\left(m_a\right) = \frac{\left(m_x^2-m_D^2\right)\left(m_x^2-m_S^2\right)}{m_x^2-m_a^2}\ , \tag{B6}$$

$$R_x\left(m_a,m_b\right) = \frac{\left(m_x^2-m_D^2\right)\left(m_x^2-m_S^2\right)}{\left(m_x^2-m_a^2\right)\left(m_x^2-m_b^2\right)}\ . \tag{B7}$$

Appendix C: ONE-LOOP WAVEFUNCTION AND MASS RENORMALIZATIONS

The necessary wavefunction renormalizations needed for the one-loop amplitudes are given in the 2+1 flavor case by

$$\delta Z_{xz} = -2\frac{\Delta f_{xz}}{f}+\frac{4}{3}\left(\frac{\Delta f_{xz}}{f}\right)_{\rm logs}, \tag{C1}$$

$$\delta Z_X = -2\frac{\Delta f_X}{f}+\frac{4}{3}\left(\frac{\Delta f_X}{f}\right)_{\rm logs}, \tag{C2}$$

where we have separated the terms in this way because the first term on the right-hand side of each of these equations is the one-loop correction to the bare decay constant f appearing in the tree-level expression for a given weak matrix element. This first term contains both NLO logarithmic corrections and Gasser-Leutwyler constants. It may be useful in chiral fits to lattice data to absorb this correction to the decay constant into the tree-level expression, and

the above formulas make this convenient. The second term is proportional to the logarithmic corrections to the decay constant alone, without the Gasser-Leutwyler constants,

$$\begin{aligned}\left(\frac{\Delta f_{xz}}{f}\right)_{\rm logs} &= \frac{1}{2f^2}\Bigg[-\left[2\ell\left(m_{xd}^2\right)+2\ell\left(m_{zd}^2\right)+\ell\left(m_{xs}^2\right)+\ell\left(m_{zs}^2\right)\right] \\ &\quad +\frac{1}{3}\Bigg(\frac{\partial R_X\left(m_\eta\right)}{\partial m_X^2}\ell(m_X^2)-R_X(m_\eta)\tilde{\ell}(m_X^2)+\frac{\partial R_\eta\left(m_X\right)}{\partial m_X^2}\ell(m_\eta^2) \\ &\quad +\frac{\partial R_Z\left(m_\eta\right)}{\partial m_Z^2}\ell(m_Z^2)-R_Z(m_\eta)\tilde{\ell}(m_Z^2)+\frac{\partial R_\eta\left(m_Z\right)}{\partial m_Z^2}\ell(m_\eta^2) \\ &\quad -2R_X\left(m_Z,m_\eta\right)\ell(m_X^2)-2R_Z\left(m_X,m_\eta\right)\ell(m_Z^2) \\ &\quad -2R_\eta\left(m_X,m_Z\right)\ell(m_\eta^2)\Bigg)\Bigg], \end{aligned} \tag{C3}$$

so that [3],

$$\frac{\Delta f_{xz}}{f} = \left(\frac{\Delta f_{xz}}{f}\right)_{\rm logs}+\frac{8}{f^2}L_4\left(2m_D^2+m_S^2\right)+\frac{8}{f^2}L_5 m_{xz}^2\ . \tag{C4}$$

For the degenerate mass case, these reduce to

$$\left(\frac{\Delta f_X}{f}\right)_{\rm logs} = \frac{1}{f^2}\left[-2\ell\left(m_{xd}^2\right)-\ell\left(m_{xs}^2\right)\right], \tag{C5}$$

$$\frac{\Delta f_X}{f} = \left(\frac{\Delta f_X}{f}\right)_{\rm logs}+\frac{8}{f^2}L_4\left(2m_D^2+m_S^2\right)+\frac{8}{f^2}L_5 m_X^2\ . \tag{C6}$$

Additionally, we need the one-loop corrections to the meson masses squared [3],

$$\begin{aligned}\frac{(\Delta m_{xz})^2}{m_{xz}^2} &= \frac{2}{3f^2}\Big(R_X(m_Z,m_\eta)\ \ell(m_X^2)+R_Z(m_X,m_\eta)\ \ell(m_Z^2)+R_\eta(m_X,m_Z)\ \ell(m_\eta^2)\Big) \\ &\quad +\frac{16}{f^2}\left(2L_8-L_5\right)m_{xz}^2+\frac{16}{f^2}\left(2L_6-L_4\right)\left(2m_D^2+m_S^2\right), \end{aligned} \tag{C7}$$

$$\begin{aligned}\frac{(\Delta m_X)^2}{m_X^2} &= \frac{2}{3f^2}\Big(-R_X(m_\eta)\ \tilde{\ell}(m_X^2)+\frac{\partial R_X(m_\eta)}{\partial m_X^2}\ \ell(m_X^2)+R_\eta(m_X,m_X)\ \ell(m_\eta^2)\Big) \\ &\quad +\frac{16}{f^2}\left(2L_8-L_5\right)m_X^2+\frac{16}{f^2}\left(2L_6-L_4\right)\left(2m_D^2+m_S^2\right). \end{aligned} \tag{C8}$$

Appendix D: LOGARITHMIC CONTRIBUTION TO $(3,\bar{3})$ $K\to\pi$ MATRIX ELEMENTS

The logarithmic contribution to the $(3,\overline{3})$, $K\to\pi$ matrix element for the 2+1 non-degenerate case is,

$$
\begin{aligned}
\left\langle \pi^+ \left| \Theta^{(3,\bar{3})} \right| K^+ \right\rangle_{\rm logs} = & - \frac{2}{f^2}\alpha^{(3,\bar{3})} \left[\frac{1}{2}\delta Z_{xz} + \frac{1}{2}\delta Z_X\right] \\
& + \frac{2}{9}\frac{\alpha^{(3,\bar{3})}}{f^4}\Bigg\{ - \frac{9m_X(2\ell(m_{xd}^2) + \ell(m_{xs}^2))}{m_{xz} - m_X} \\
& + \left[-2 + \frac{3m_X}{m_{xz} - m_X} + \left(2 - \frac{3m_X}{m_{xz} - m_X}\right)R_\eta(m_X, m_X) \right. \\
& \left. - 2R_X(m_Z, m_\eta)\right]\ell(m_X^2) \\
& + \left[-2 - \frac{3m_X}{m_{xz} - m_X} + \left(2 + \frac{3m_X}{m_{xz} - m_X}\right)R_\eta(m_Z, m_Z) \right. \\
& \left. - 2R_Z(m_X, m_\eta)\right]\ell(m_Z^2) \\
& + 6\left[2 + \frac{3m_X}{m_{xz} - m_X}\right]\ell(m_{zd}^2) + 3\left[2 + \frac{3m_X}{m_{xz} - m_X}\right]\ell(m_{zs}^2) \\
& + \left[\left(-2 + \frac{3m_X}{m_{xz} - m_X}\right)R_\eta(m_X, m_X) - 2R_\eta(m_X, m_Z) \right. \\
& \left. - \left(2 + \frac{3m_X}{m_{xz} - m_X}\right)R_\eta(m_Z, m_Z)\right]\ell(m_\eta^2)
\end{aligned}
$$

$$
\begin{aligned}
&-\left[\left(2+\frac{3m_X}{m_{xz}-m_X}\right)R_X(m_\eta)+2\left(2m_{xz}^2+m_X^2\right)R_X(m_Z,m_\eta)\right]\beta(q^2,m_{xz}^2,m_X^2)\\
&+\left[2m_Z^2-4m_{xz}^2+\frac{3m_X(m_Z^2-m_X^2)}{m_{xz}-m_X}-4m_X^2+\left(2+\frac{3m_X}{m_{xz}-m_X}\right)R_Z(m_\eta)\right.\\
&+\left(-2m_Z^2+4m_{xz}^2+\frac{3m_X(-m_Z^2+m_X^2)}{m_{xz}-m_X}+4m_X^2\right)R_\eta(m_Z,m_Z)\\
&\left.-2\left(2m_{xz}^2+m_X^2\right)R_Z(m_X,m_\eta)\right]\beta(q^2,m_Z^2,m_{xz}^2)\\
&+6\left[-2m_{zd}^2+2m_{xd}^2+2m_{xz}^2+3m_X\left(\frac{-m_{zd}^2+m_{xd}^2}{m_{xz}-m_X}-m_{xz}\right)+m_X^2\right]\beta(q^2,m_{zd}^2,m_{xd}^2)\\
&+3\left[-2m_{zs}^2+2m_{xs}^2+2m_{xz}^2+3m_X\left(\frac{-m_{zs}^2+m_{xs}^2}{m_{xz}-m_X}-m_{xz}\right)+m_X^2\right]\beta(q^2,m_{zs}^2,m_{xs}^2)\\
&+\left[\left(-2m_\eta^2+\frac{3m_X(-m_\eta^2+m_X^2)}{m_{xz}-m_X}+2m_X^2\right)R_\eta(m_X,m_X)\right.\\
&-2\left(2m_{xz}^2+m_X^2\right)R_\eta(m_X,m_Z)\\
&\left.+\left(2m_\eta^2-4m_{xz}^2+\frac{3m_X(m_\eta^2-m_X^2)}{m_{xz}-m_X}-4m_X^2\right)R_\eta(m_Z,m_Z)\right]\beta(q^2,m_\eta^2,m_{xz}^2)\\
&+\left[2-\frac{3m_X}{m_{xz}-m_X}\right]R_X(m_\eta)\tilde{\ell}(m_X^2)+\left[2+\frac{3m_X}{m_{xz}-m_X}\right]R_Z(m_\eta)\tilde{\ell}(m_Z^2)\\
&+\left[2m_Z^2-4m_{xz}^2+\frac{3m_X(m_Z^2-m_X^2)}{m_{xz}-m_X}-4m_X^2\right]R_Z(m_\eta)\tilde{\beta}(q^2,m_Z^2,m_{xz}^2)\Bigg\}
\end{aligned}
\tag{D1}
$$

Appendix E: LOGARITHMIC CONTRIBUTION TO (8,8) $K \to \pi$ MATRIX ELEMENTS

The logarithmic contribution to the (8,8), $\Delta I = 3/2$, $K \to \pi$ matrix element in the 2+1 non-degenerate case is

$$
\begin{aligned}
\left\langle \pi^+ \left| \mathcal{O}^{(8,8)(3/2)} \right| K^+ \right\rangle_{Q_1,\mathrm{logs}} = {} & \frac{4\alpha_{88}}{f^2}\left(\frac{1}{2}\delta Z_X + \frac{1}{2}\delta Z_{xz}\right) \\
& + \frac{8}{9}\frac{\alpha_{88}}{f^4}\Bigg\{\Big[1 - 2R_X(m_Z, m_\eta) - R_\eta(m_X, m_X)\Big]\ell(m_X^2) \\
& + \Big[1 - 2R_Z(m_X, m_\eta) - R_\eta(m_Z, m_Z)\Big]\ell(m_Z^2) \\
& - 18\ell(m_{xd}^2) - 9\ell(m_{xs}^2) - 6\ell(m_{zd}^2) - 3\ell(m_{zs}^2) \\
& + \Big[R_\eta(m_Z, m_Z) - 2R_\eta(m_X, m_Z) + R_\eta(m_X, m_X)\Big]\ell(m_\eta^2) \\
& - R_X(m_\eta)\tilde{\ell}(m_X^2) - R_Z(m_\eta)\tilde{\ell}(m_Z^2) \\
& - 9m_{xz}m_X\beta(q^2, m_{xz}^2, m_X^2)\Bigg\}.
\end{aligned} \tag{E1}
$$

The logarithmic contribution to the (8,8), $\Delta I = 1/2$, $K \to \pi$ matrix element in the 2+1 non-degenerate case is

$$
\begin{aligned}
\left\langle \pi^+ \left| \mathcal{O}^{(8,8)(1/2)} \right| K^+ \right\rangle_{Q_1,\mathrm{logs}} = {} & \frac{8\alpha_{88}}{f^2}\left(\frac{1}{2}\delta Z_X + \frac{1}{2}\delta Z_{xz}\right) \\
& + \frac{2}{9}\frac{\alpha_{88}}{f^4}\Bigg\{8\Big[1 - 2R_X(m_Z, m_\eta) - R_\eta(m_X, m_X)\Big]\ell(m_X^2) \\
& + 8\Big[1 - 2R_Z(m_X, m_\eta) - R_\eta(m_Z, m_Z)\Big]\ell(m_Z^2) \\
& + 18\frac{4m_X - 3m_{xz}}{m_{xz} - m_X}\left[2\ell(m_{xd}^2) + \ell(m_{xs}^2)\right] \\
& + 6\frac{4m_X - 7m_{xz}}{m_{xz} - m_X}\left[2\ell(m_{zd}^2) + \ell(m_{zs}^2)\right] \\
& + 8\Big[R_\eta(m_Z, m_Z) - 2R_\eta(m_X, m_Z) + R_\eta(m_X, m_X)\Big]\ell(m_\eta^2) \\
& - 8R_X(m_\eta)\tilde{\ell}(m_X^2) - 8R_Z(m_\eta)\tilde{\ell}(m_Z^2) \\
& + 36m_{xz}m_X\beta(q^2, m_{xz}^2, m_X^2) \\
& + 36m_X m_{xz}\left(2\beta(q^2, m_{zd}^2, m_{xd}^2) + \beta(q^2, m_{zs}^2, m_{xs}^2)\right)\Bigg\}.
\end{aligned} \tag{E2}
$$

For completeness we include the chiral corrections to $K \to \pi\pi$ at physical kinematics for the electro-weak penguin operators. In the full theory at physical kinematics, the logarithmic

contribution to the (8,8), $\Delta I = 3/2$, $K \to \pi\pi$ amplitude is

$$\begin{aligned}\langle \pi^+\pi^- | \mathcal{O}^{(8,8),(3/2)} | K^0 \rangle_{\rm logs} \; = \; & -4i \frac{\alpha_{88}}{f_K f_\pi^2 f^2} \left[\left(\frac{5 m_K^4}{4 m_\pi^2} - 2 m_K^2 \right) \beta(m_\pi^2, m_K^2, m_\pi^2) \right. \\ & + (m_K^2 - 2 m_\pi^2) \beta(m_K^2, m_\pi^2, m_\pi^2) \\ & + \frac{m_K^4}{4 m_\pi^2} \beta(m_\pi^2, m_K^2, m_\eta^2) - \left(4 + \frac{m_K^2}{2 m_\pi^2} \right) \ell(m_K^2) \\ & \left. + \left(\frac{5 m_K^2}{4 m_\pi^2} - 8 \right) \ell(m_\pi^2) - \frac{3 m_K^2}{4 m_\pi^2} \ell(m_\eta^2) \right], \end{aligned} \tag{E3}$$

and the logarithmic contribution to the (8,8), $\Delta I = 1/2$, $K \to \pi\pi$ amplitude is

$$\begin{aligned}\langle \pi^+\pi^- | \mathcal{O}^{(8,8),(1/2)} | K^0 \rangle_{\rm logs} \; = \; & -8i \frac{\alpha_{88}}{f_K f_\pi^2 f^2} \left[\left(\frac{m_K^4}{2 m_\pi^2} - 2 m_K^2 \right) \beta(m_\pi^2, m_K^2, m_\pi^2) \right. \\ & + \frac{3}{4} m_K^2 \beta(m_K^2, m_K^2, m_K^2) + (m_\pi^2 - 2 m_K^2) \beta(m_K^2, m_\pi^2, m_\pi^2) \\ & + \frac{m_K^4}{4 m_\pi^2} \beta(m_\pi^2, m_K^2, m_\eta^2) + \frac{1}{4} \left(\frac{m_K^2}{m_\pi^2} - 22 \right) \ell(m_K^2) \\ & \left. + \frac{1}{4} \left(\frac{2 m_K^2}{m_\pi^2} - 26 \right) \ell(m_\pi^2) - \frac{3 m_K^2}{4 m_\pi^2} \ell(m_\eta^2) \right]. \end{aligned} \tag{E4}$$

Appendix F: LOGARITHMIC CONTRIBUTION TO (27,1) $K \to \pi$ MATRIX ELEMENTS

The logarithmic contribution to the (27,1), $\Delta I = 3/2$, $K \to \pi$ matrix element in the 2+1 non-degenerate case is

$$\begin{aligned}\left\langle \pi^+ \left| \mathcal{O}^{(27,1)(3/2)} \right| K^+ \right\rangle_{\rm logs} \; = \; & -\frac{4\alpha_{27}}{f^2} m_X m_{xz} \left[\frac{1}{2} \delta Z_X + \frac{1}{2} \delta Z_{xz} + \frac{1}{2} \left(\frac{\delta m_X^2}{m_X^2} + \frac{\delta m_{xz}^2}{m_{xz}^2} \right) \right] \\ & + \frac{8}{9} \frac{\alpha_{27}}{f^4} m_X m_{xz} \Bigg\{ \Big[8 + R_\eta(m_X, m_X) + 2 R_X(m_Z, m_\eta) \Big] \ell(m_X^2) \\ & + \Big[-1 + R_\eta(m_Z, m_Z) + 2 R_Z(m_X, m_\eta) \Big] \ell(m_Z^2) \\ & + 9 \ell(m_{xz}^2) + 18 \ell(m_{xd}^2) + 9 \ell(m_{xs}^2) + 6 \ell(m_{zd}^2) + 3 \ell(m_{zs}^2) \\ & + \Big[-R_\eta(m_X, m_X) + 2 R_\eta(m_X, m_Z) - R_\eta(m_Z, m_Z) \Big] \ell(m_\eta^2) \\ & + R_X(m_\eta) \tilde{\ell}(m_X^2) + R_Z(m_\eta) \tilde{\ell}(m_Z^2) + 9 m_X m_{xz} \beta(q^2, m_{xz}^2, m_X^2) \Bigg\}. \end{aligned} \tag{F1}$$

The logarithmic contribution to the (27,1), $\Delta I = 1/2$, $K \to \pi$ matrix element in the 2+1 non-degenerate case is

$$
\begin{aligned}
\left\langle \pi^+ \left| \mathcal{O}^{(27,1)(1/2)} \right| K^+ \right\rangle_{\text{logs}} = & -\frac{4\alpha_{27}}{f^2} m_{xz} m_X \left[\frac{1}{2}\delta Z_X + \frac{1}{2}\delta Z_{xz} + \frac{1}{2}\left(\frac{\delta m_X^2}{m_X^2} + \frac{\delta m_{xz}^2}{m_{xz}^2} \right) \right] \\
& + \frac{2}{9}\frac{\alpha_{27}}{f^4} \Bigg\{ 12 m_{xz} m_X \left(6\ell(m_{xd}^2) + 3\ell(m_{xs}^2) + 2\ell(m_{zd}^2) + \ell(m_{zs}^2) \right) \\
& + \Bigg[\Bigg(12 m_\eta^2 - 4 m_{xz} m_X + 6 m_X^2 \\
& + \frac{9 m_X (m_\eta^2 + m_X^2)}{m_{xz} - m_X} \Bigg) R_\eta(m_X, m_X) \\
& - 2 \Bigg(12 m_\eta^2 + 6 m_{xz}^2 - 4 m_{xz} m_X + 9 m_X^2 \\
& + \frac{9 m_X (m_\eta^2 + m_X^2)}{m_{xz} - m_X} \Bigg) R_\eta(m_X, m_Z) \\
& + \Bigg(12 m_\eta^2 + 12 m_{xz}^2 - 4 m_{xz} m_X + 12 m_X^2 \\
& + \frac{9 m_X (m_\eta^2 + m_X^2)}{m_{xz} - m_X} \Bigg) R_\eta(m_Z, m_Z) \Bigg] \ell(m_\eta^2) \Bigg\} \\
& + \frac{4\alpha_{27} m_{xz}}{f^4} \left[5 m_X + \frac{6 m_{xz}^2}{m_{xz} - m_X} \right] \ell(m_{xz}^2)
\end{aligned}
$$

$$
\begin{aligned}
& + \frac{2}{9}\frac{\alpha_{27}}{f^4} \Bigg\{ \Bigg[\frac{4 m_X m_{xz} (10 m_X - 19 m_{xz})}{m_{xz} - m_X} + 3 \frac{4 m_{xz} - m_X}{m_{xz} - m_X} R_X(m_\eta) \\
& + 2 \frac{m_X m_{xz} (2 m_{xz} - 11 m_X)}{m_{xz} - m_X} R_\eta(m_X, m_X) \\
& + 2 \left(-6 m_{xz}^2 + 4 m_{xz} m_X - 21 m_X^2 - \frac{18 m_X^3}{m_{xz} - m_X} \right) R_X(m_Z, m_\eta) \Bigg] \ell(m_X^2) \\
& + \Bigg[-72 m_{xz}^2 - 4 m_{xz} m_X - \frac{36 m_X m_{xz}^2}{m_{xz} - m_X} + 3 \frac{4 m_{xz} - m_X}{m_{xz} - m_X} R_Z(m_\eta) \\
& + \left(-36 m_{xz}^2 + 4 m_{xz} m_X - \frac{18 m_X m_{xz}^2}{m_{xz} - m_X} \right) R_\eta(m_Z, m_Z) \\
& + 2 \left(-15 m_Z^2 + 4 m_{xz} m_X - 12 m_X^2 - \frac{18 m_X m_{xz}^2}{m_{xz} - m_X} \right) R_Z(m_X, m_\eta) \Bigg] \ell(m_Z^2) \\
& + \frac{2 m_X m_{xz} (2 m_{xz} - 11 m_X)}{m_{xz} - m_X} R_X(m_\eta) \tilde{\ell}(m_X^2) \\
& + \left[4 m_{xz} m_X - 36 m_{xz}^2 - \frac{18 m_X m_{xz}^2}{m_{xz} - m_X} \right] R_Z(m_\eta) \tilde{\ell}(m_Z^2) \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{3}\frac{\alpha_{27}}{f^4}\Bigg\{\Bigg[-2m_\eta^4-4m_\eta^2 m_{xz}m_X-2m_\eta^2 m_X^2+4m_{xz}m_X^3+4m_X^4\\
&+\frac{3m_X(m_X^4-m_\eta^4)}{m_{xz}-m_X}\Bigg]R_\eta(m_X,m_X)\\
&+2\Bigg[2m_\eta^4-2m_\eta^2 m_{xz}^2+4m_\eta^2 m_{xz}m_X-4m_{xz}^3 m_X+m_\eta^2 m_X^2+2m_{xz}^2 m_X^2\\
&-6m_{xz}m_X^3-3m_X^4+\frac{3m_X(m_\eta^4-m_X^4)}{m_{xz}-m_X}\Bigg]R_\eta(m_X,m_Z)\\
&+\Bigg[-2m_\eta^4+4m_\eta^2 m_{xz}^2-4m_\eta^2 m_{xz}m_X+8m_{xz}^3 m_X-4m_{xz}^2 m_X^2+8m_{xz}m_X^3\\
&+2m_X^4+\frac{3m_X(m_X^4-m_\eta^4)}{m_{xz}-m_X}\Bigg]R_\eta(m_Z,m_Z)\Bigg\}\beta(q^2,m_\eta^2,m_{xz}^2)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{3}\frac{\alpha_{27}}{f^4}\Bigg\{-2m_X m_{xz}\Bigg[12m_{xz}m_X-\frac{m_X+2m_{xz}}{m_X-m_{xz}}R_X(m_\eta)\\
&+2\Big(2m_{xz}^2+m_X^2\Big)R_X(m_Z,m_\eta)\Bigg]\beta(q^2,m_{xz}^2,m_X^2)\\
&+\Bigg[6m_X\left(-4m_X^2 m_{xz}+\frac{m_Z^4-m_X^4}{m_{xz}-m_X}\right)+2\Bigg(2m_{xz}^2-2m_Z^2-2m_{xz}m_X\\
&-\frac{3m_Z^2 m_X}{m_{xz}-m_X}\Bigg)R_Z(m_\eta)\\
&+3m_X\left(-4m_X^2 m_{xz}+\frac{m_Z^4-m_X^4}{m_{xz}-m_X}\right)R_\eta(m_Z,m_Z)\\
&+2\Bigg(m_Z^4-2m_X^3(m_X+4m_{xz})+m_X m_Z^2(m_X+2m_{xz})\\
&+\frac{3m_X(m_Z^4-m_X^4)}{m_{xz}-m_X}\Bigg)R_Z(m_X,m_\eta)\Bigg]\beta(q^2,m_{xz}^2,m_Z^2)\\
&+3m_X\left[4m_X^2 m_{xz}+\frac{m_X^4-m_Z^4}{m_{xz}-m_X}\right]R_Z(m_\eta)\tilde{\beta}(q^2,m_Z^2,m_{xz}^2)\Bigg\}.
\end{aligned}
\tag{F2}
$$

Appendix G: LOGARITHMIC CONTRIBUTION TO (8,1) $K \to 0$ AND $K \to \pi$ MATRIX ELEMENTS

For the (8,1) case, we separate the logarithm terms which are proportional to α_1 from those proportional to α_2.

For $K \to 0$, we have

$$
\left\langle 0\left|\mathcal{O}^{(8,1)}\right|K^0\right\rangle_{\rm logs}=\left\langle 0\left|\mathcal{O}^{(8,1)}\right|K^0\right\rangle_{\rm logs}^{(1)}+\left\langle 0\left|\mathcal{O}^{(8,1)}\right|K^0\right\rangle_{\rm logs}^{(2)}\ , \tag{G1}
$$

where the superscripts refer to the terms proportional to $\alpha_{1,2}$, and

$$\begin{aligned}\left\langle 0 \left| \mathcal{O}^{(8,1)} \right| K^0 \right\rangle^{(1)}_{\rm logs} \;=\; & \frac{4i}{3f^3}\alpha_1 \Bigg\{ \Big[m_X^2 + R_X(m_\eta) - m_X^2 R_\eta(m_X,m_X) \Big] \ell(m_X^2) \\ & + \Big[-m_Z^2 - R_Z(m_\eta) + m_Z^2 R_\eta(m_Z,m_Z) \Big] \ell(m_Z^2) \\ & + m_\eta^2 \Big[R_\eta(m_X,m_X) - R_\eta(m_Z,m_Z) \Big] \ell(m_\eta^2) \\ & - 6 m_{xd}^2 \ell(m_{xd}^2) - 3 m_{xs}^2 \ell(m_{xs}^2) + 6 m_{zd}^2 \ell(m_{zd}^2) + 3 m_{zs}^2 \ell(m_{zs}^2) \\ & - m_X^2 R_X(m_\eta) \tilde{\ell}(m_X^2) + m_Z^2 R_Z(m_\eta) \tilde{\ell}(m_Z^2) \Bigg\}, \end{aligned} \tag{G2}$$

$$\begin{aligned}\left\langle 0 \left| \mathcal{O}^{(8,1)} \right| K^0 \right\rangle^{(2)}_{\rm logs} \;=\; & \frac{4i}{f}\alpha_2 (m_{xz}^2 - m_X^2) \left[\frac{1}{2} \delta Z_{xz} \right] \\ & + \frac{8i}{9}\frac{\alpha_2}{f^3}(m_{xz}^2 - m_X^2) \Bigg\{ \Big[1 + R_X(m_\eta,m_Z) - R_\eta(m_X,m_X) \Big] \ell(m_X^2) \\ & + \Big[1 + R_Z(m_\eta,m_X) - R_\eta(m_Z,m_Z) \Big] \ell(m_Z^2) \\ & + \Big[R_\eta(m_X,m_X) + R_\eta(m_X,m_Z) + R_\eta(m_Z,m_Z) \Big] \ell(m_\eta^2) \\ & - 6\ell(m_{xd}^2) - 3\ell(m_{xs}^2) - 6\ell(m_{zd}^2) - 3\ell(m_{zs}^2) \\ & - R_X(m_\eta)\tilde{\ell}(m_X^2) - R_Z(m_\eta)\tilde{\ell}(m_Z^2) \Bigg\}. \end{aligned} \tag{G3}$$

For $K \to \pi$, we again separate the logarithmic terms proportional to α_1 and α_2,

$$\left\langle \pi^+ \left| \mathcal{O}^{(8,1)} \right| K^+ \right\rangle_{\rm logs} = \left\langle \pi^+ \left| \mathcal{O}^{(8,1)} \right| K^+ \right\rangle^{(1)}_{\rm logs} + \left\langle \pi^+ \left| \mathcal{O}^{(8,1)} \right| K^+ \right\rangle^{(2)}_{\rm logs} \ , \tag{G4}$$

with

$$
\begin{aligned}
\left\langle \pi^+ \left| \mathcal{O}^{(8,1)} \right| K^+ \right\rangle^{(1)}_{\rm logs} \;=\;& \frac{4\alpha_1}{f^2} m_{xz} m_X \left[\frac{1}{2}\delta Z_X + \frac{1}{2}\delta Z_{xz} + \frac{1}{2}\left(\frac{\delta m_X^2}{m_X^2} + \frac{\delta m_{xz}^2}{m_{xz}^2} \right) \right] \\
&+ \frac{1}{9}\frac{\alpha_1}{f^4} \Bigg\{ 36 \left[2m_{xd}^2 - m_X^2 - m_{xz} m_X + \frac{m_X(m_{zd}^2 + m_{xd}^2)}{m_{xz} - m_X} \right] \ell(m_{xd}^2) \\
&+ 18 \left[2m_{xs}^2 - m_X^2 - m_{xz} m_X + \frac{m_X(m_{zs}^2 + m_{xs}^2)}{m_{xz} - m_X} \right] \ell(m_{xs}^2) \\
&- 12 \left[6m_{zd}^2 + 3m_X^2 + m_{xz} m_X + \frac{3m_X(m_{zd}^2 + m_{xd}^2)}{m_{xz} - m_X} \right] \ell(m_{zd}^2) \\
&- 6 \left[6m_{zs}^2 + 3m_X^2 + m_{xz} m_X + \frac{3m_X(m_{zs}^2 + m_{xs}^2)}{m_{xz} - m_X} \right] \ell(m_{zs}^2) \\
&+ 2 \Bigg[\left(-4m_\eta^2 + 4m_{xz} m_X - 2m_X^2 - \frac{3m_X(m_\eta^2 + m_X^2)}{m_{xz} - m_X} \right) R_\eta(m_X, m_X) \\
&+ 2 \left(2m_{xz}^2 - 4m_{xz} m_X + m_X^2 \right) R_\eta(m_X, m_Z) \\
&+ \Bigg(4m_\eta^2 + 4m_{xz}^2 + 4m_{xz} m_X + 4m_X^2 \\
&+ \frac{3m_X(m_\eta^2 + m_X^2)}{m_{xz} - m_X} \Bigg) R_\eta(m_Z, m_Z) \Bigg] \ell(m_\eta^2) \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{9}\frac{\alpha_1}{f^4}\Bigg\{\Bigg[2m_X m_{xz}\frac{5m_X-2m_{xz}}{m_{xz}-m_X}\left(R_\eta(m_X,m_X)-1\right)-\frac{4m_{xz}-m_X}{m_{xz}-m_X}R_X(m_\eta)\\
&+2\Big(2m_{xz}^2-4m_{xz}m_X+m_X^2\Big)R_X(m_Z,m_\eta)\Bigg]\ell(m_X^2)\\
&+\Bigg[\frac{4m_{xz}-m_X}{m_{xz}-m_X}R_Z(m_\eta)-\Bigg(4m_{xz}(3m_{xz}+m_X)+\frac{6m_X m_{xz}^2}{m_{xz}-m_X}\Bigg)\left(R_\eta(m_Z,m_Z)-1\right)\\
&+2\Big(2m_{xz}^2-4m_{xz}m_X+m_X^2\Big)R_Z(m_X,m_\eta)\Bigg]\ell(m_Z^2)\Bigg\}\\
&+\frac{2}{9}\frac{\alpha_1}{f^4}\Bigg\{-12m_{xz}m_X\ell(m_{xz}^2)+2m_X\Bigg[3m_X-2m_{xz}+\frac{3m_X^2}{m_{xz}-m_X}\Bigg]R_X(m_\eta)\tilde{\ell}(m_X^2)\\
&-2m_{xz}\Bigg[2(3m_{xz}+m_X)+\frac{3m_X m_{xz}}{m_{xz}-m_X}\Bigg]R_Z(m_\eta)\tilde{\ell}(m_Z^2)\Bigg\}\\
&+\frac{1}{9}\frac{\alpha_1}{f^4}\Bigg\{36\Bigg[m_X^2(4m_{xz}^2-m_{xd}^2-m_{zd}^2)+(m_{xd}^2+m_{zd}^2-2m_X^2-2m_{xz}^2)m_{xz}m_X\\
&+\frac{m_X(m_{zd}^4-m_{xd}^4)}{m_{xz}-m_X}\Bigg]\beta(q^2,m_{zd}^2,m_{xd}^2)\\
&+18\Bigg[m_X^2(4m_{xz}^2-m_{xs}^2-m_{zs}^2)+(m_{xs}^2+m_{zs}^2-2m_{xz}^2-2m_X^2)m_{xz}m_X\\
&+\frac{m_X(m_{zs}^4-m_{xs}^4)}{m_{xz}-m_X}\Bigg]\beta(q^2,m_{zs}^2,m_{xs}^2)\\
&+2\Bigg[\Bigg(2m_\eta^4+4m_\eta^2 m_{xz}m_X+2m_\eta^2 m_X^2-4m_{xz}m_X^3-4m_X^4\\
&+\frac{3m_X(m_\eta^4-m_X^4)}{m_{xz}-m_X}\Bigg)R_\eta(m_X,m_X)\\
&+2\Big(2m_\eta^2 m_{xz}^2+4m_{xz}^3 m_X+m_\eta^2 m_X^2-2m_{xz}^2 m_X^2+2m_{xz}m_X^3-m_X^4\Big)R_\eta(m_X,m_Z)\\
&+\Bigg(-2m_\eta^4+4m_\eta^2 m_{xz}^2-4m_\eta^2 m_{xz}m_X+8m_{xz}^3 m_X-4m_{xz}^2 m_X^2+8m_{xz}m_X^3\\
&+2m_X^4+\frac{3m_X(m_X^4-m_\eta^4)}{m_{xz}-m_X}\Bigg)R_\eta(m_Z,m_Z)\Bigg]\beta(q^2,m_\eta^2,m_{xz}^2)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{9}\frac{\alpha_1}{f^4}\Bigg\{2m_X m_{xz}\Bigg[\frac{2m_{xz}+m_X}{m_{xz}-m_X}R_X(m_\eta)\\
&+2\Big(2m_{xz}^2+m_X^2\Big)R_X(m_Z,m_\eta)\Bigg]\beta(q^2,m_{xz}^2,m_X^2)\\
&+\Bigg[12m_X^3 m_{xz}+\frac{3m_X(m_X^4-m_Z^4)}{m_{xz}-m_X}+2\Bigg(-2m_{xz}^2-2m_{xz}m_X+2m_X^2\\
&-\frac{3m_Z^2 m_X}{m_{xz}-m_X}\Bigg)R_Z(m_\eta)\\
&+2\Big(m_Z^4-2m_X^4+m_X^2 m_Z^2+2(2m_{xz}^2+m_X^2)m_X m_{xz}\Big)R_Z(m_X,m_\eta)\\
&+\Bigg(-12m_{xz}m_X^3+\frac{3m_X(m_Z^4-m_X^4)}{m_{xz}-m_X}\Bigg)R_\eta(m_Z,m_Z)\Bigg]\beta(q^2,m_Z^2,m_{xz}^2)\\
&+\Bigg[4(2m_X^2+2m_{xz}^2-m_Z^2)m_{xz}m_X\\
&+\frac{3m_X(m_X^4-m_Z^4)}{m_{xz}-m_X}\Bigg]R_Z(m_\eta)\tilde{\beta}(q^2,m_Z^2,m_{xz}^2)\Bigg\},
\end{aligned}
\tag{G5}
$$

and

$$
\begin{aligned}
\left\langle\pi^+\left|\mathcal{O}^{(8,1)}\right|K^+\right\rangle_{\text{logs}}^{(2)}=&-\frac{4\alpha_2}{f^2}m_{xz}^2\left[\frac{1}{2}\delta Z_X+\frac{1}{2}\delta Z_{xz}\right]\\
&+\frac{4}{9}\frac{\alpha_2}{f^4}m_{xz}^2\Bigg\{-\frac{9m_X(2\ell(m_{xd}^2)+\ell(m_{xs}^2))}{m_{xz}-m_X}\\
&+\Bigg[-2+\frac{3m_X}{m_{xz}-m_X}+\Bigg(2-\frac{3m_X}{m_{xz}-m_X}\Bigg)R_\eta(m_X,m_X)\\
&-2R_X(m_Z,m_\eta)\Bigg]\ell(m_X^2)\\
&+\Bigg[-2-\frac{3m_X}{m_{xz}-m_X}+\Bigg(2+\frac{3m_X}{m_{xz}-m_X}\Bigg)R_\eta(m_Z,m_Z)\\
&-2R_Z(m_X,m_\eta)\Bigg]\ell(m_Z^2)\\
&+6\Bigg[2+\frac{3m_X}{m_{xz}-m_X}\Bigg]\ell(m_{zd}^2)+3\Bigg[2+\frac{3m_X}{m_{xz}-m_X}\Bigg]\ell(m_{zs}^2)\\
&+\Bigg[\Bigg(-2+\frac{3m_X}{m_{xz}-m_X}\Bigg)R_\eta(m_X,m_X)-2R_\eta(m_X,m_Z)\\
&-\Bigg(2+\frac{3m_X}{m_{xz}-m_X}\Bigg)R_\eta(m_Z,m_Z)\Bigg]\ell(m_\eta^2)
\end{aligned}
$$

$$
\begin{aligned}
&- \left[\left(2+\frac{3m_X}{m_{xz}-m_X}\right)R_X(m_\eta)+2\left(2m_{xz}^2+m_X^2\right)R_X(m_Z,m_\eta)\right]\beta(q^2,m_{xz}^2,m_X^2)\\
&+ \left[2m_Z^2-4m_{xz}^2+\frac{3m_X(m_Z^2-m_X^2)}{m_{xz}-m_X}-4m_X^2+\left(2+\frac{3m_X}{m_{xz}-m_X}\right)R_Z(m_\eta)\right.\\
&+ \left(-2m_Z^2+4m_{xz}^2+\frac{3m_X(-m_Z^2+m_X^2)}{m_{xz}-m_X}+4m_X^2\right)R_\eta(m_Z,m_Z)\\
&\left.- 2\left(2m_{xz}^2+m_X^2\right)R_Z(m_X,m_\eta)\right]\beta(q^2,m_Z^2,m_{xz}^2)\\
&+ 6\left[-2m_{zd}^2+2m_{xd}^2+2m_{xz}^2+3m_X\left(\frac{-m_{zd}^2+m_{xd}^2}{m_{xz}-m_X}-m_{xz}\right)+m_X^2\right]\beta(q^2,m_{zd}^2,m_{xd}^2)\\
&+ 3\left[-2m_{zs}^2+2m_{xs}^2+2m_{xz}^2+3m_X\left(\frac{-m_{zs}^2+m_{xs}^2}{m_{xz}-m_X}-m_{xz}\right)+m_X^2\right]\beta(q^2,m_{zs}^2,m_{xs}^2)\\
&+ \left[\left(-2m_\eta^2+\frac{3m_X(-m_\eta^2+m_X^2)}{m_{xz}-m_X}+2m_X^2\right)R_\eta(m_X,m_X)\right.\\
&- 2\left(2m_{xz}^2+m_X^2\right)R_\eta(m_X,m_Z)\\
&\left.+ \left(2m_\eta^2-4m_{xz}^2+\frac{3m_X(m_\eta^2-m_X^2)}{m_{xz}-m_X}-4m_X^2\right)R_\eta(m_Z,m_Z)\right]\beta(q^2,m_\eta^2,m_{xz}^2)\\
&+ \left[2-\frac{3m_X}{m_{xz}-m_X}\right]R_X(m_\eta)\tilde{\ell}(m_X^2)+\left[2+\frac{3m_X}{m_{xz}-m_X}\right]R_Z(m_\eta)\tilde{\ell}(m_Z^2)\\
&+ \left[2m_Z^2-4m_{xz}^2+\frac{3m_X(m_Z^2-m_X^2)}{m_{xz}-m_X}-4m_X^2\right]R_Z(m_\eta)\tilde{\beta}(q^2,m_Z^2,m_{xz}^2)\Bigg\}. \qquad \text{(G6)}
\end{aligned}
$$

These formulas are simplified enormously in the degenerate valence case,

$$
\begin{aligned}
\left\langle\pi^+\left|\mathcal{O}^{(8,1)}\right|K^+\right\rangle_{\text{logs}}^{\text{deg.val.},(1)} &= \frac{4\alpha_1}{f^2}m_X^2\left[\delta Z_X+\frac{\delta m_X^2}{m_X^2}\right]\\
&+ \frac{4}{3}\frac{\alpha_1}{f^4}m_X^2\Bigg\{2\Bigg[\ell(m_\eta^2)\\
&+ \left(m_\eta^2+m_X^2\right)\beta(0,m_\eta^2,m_X^2)\Bigg]R_\eta(m_X,m_X)\\
&+ 2\Big[2-R_\eta(m_X,m_X)\Big]\ell(m_X^2)-10\ell(m_{xd}^2)\\
&- 5\ell(m_{xs}^2)-2m_X^2R_X(m_\eta)\tilde{\tilde{\ell}}(m_X^2)\\
&+ 4\Big[-m_X^2-R_X(m_\eta)+m_X^2R_\eta(m_X,m_X)\Big]\tilde{\ell}(m_X^2)\Bigg\}, \qquad \text{(G7)}
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \pi^+ \left| \mathcal{O}^{(8,1)} \right| K^+ \right\rangle^{\text{deg.val.},(2)}_{\text{logs}} =& -\frac{4\alpha_2}{f^2} m_X^2 \delta Z_X \\
&+ \frac{8}{3}\frac{\alpha_2}{f^4} m_X^2 \Bigg\{ m_X^2 \left[R_X(m_\eta) \tilde{\tilde{\ell}}(m_X^2) - 2R_\eta(m_X, m_X)\beta(0, m_\eta^2, m_X^2) \right] \\
&+ \left[-1 + R_\eta(m_X, m_X) \right] \ell(m_X^2) + 2\ell(m_{xd}^2) + \ell(m_{xs}^2) \\
&- R_\eta(m_X, m_X)\ell(m_\eta^2) \\
&+ \left[2m_X^2 \Big(1 - R_\eta(m_X, m_X) \Big) + R_X(m_\eta) \right] \tilde{\ell}(m_X^2) \Bigg\}. \qquad \text{(G8)}
\end{aligned}
$$

Appendix H: ERRATUM

We note here some corrections to the works of Refs. [10] and [11]. All of the NLO low energy constants for the (27,1) operators have the wrong sign in both Ref. [10] and Ref. [11]. The values for γ_i appearing in Table I of Ref. [10] (and again in Table I of Ref. [11]) should have the opposite sign. In Eq. (16) of Ref. [10] (and again in Eq. (16) of Ref. [11]), the operators $\mathcal{O}_5^{(8,1)}$ and $\mathcal{O}_{15}^{(8,1)}$ should have opposite sign to be consistent with the signs of the LEC's e_5^r and e_{15}^r appearing in the amplitudes presented in those works. We make these corrections in the current work. Since the LEC's are not known, an incorrect, but consistent normalization of them (including an incorrect sign) does not alter the procedure of using the formulas of Ref. [11] to construct $K \to \pi\pi$ from $K \to \pi$ and $K \to 0$ matrix elements. Therefore, these corrections make no difference to the conclusions of these works that it is possible to obtain all of the LEC's needed to construct $K \to \pi\pi$ matrix elements through NLO in χPT from lattice calculations.

Additionally in Ref. [11], there is a typo in the (8,8) $K \to \pi\pi$ matrix element formulas. The coefficient of the second term in Eq. (42) should be $\frac{12i}{f_K f_\pi^2}$, and the coefficient of the second term in Eq. (43) should be $-\frac{12i}{f_K f_\pi^2}$. The corrected versions of these equations are given in the current work as Eqs. (38) and (39). These corrections also do not alter the conclusions of Ref. [11], but are necessary to construct the correct $K \to \pi\pi$ matrix elements.

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